PART - A

Unit - I

DIFFERENTIAL CALCULUS - 1

1.1 Successive Differentiation

1.11 Introduction

As we all know, the meaning of the word 'successive' is one after another or again and again. This topic deals with differentiation of a given function of a single variable again and again.

1.12 Successive (Higher order) Derivatives

If $y = f(x)$ we know that $\frac{dy}{dx} = f'(x)$ is called as the derivative of y $w.r.t$ x. As we have the objective of differentiating again and again this shall be called as the first derivative of y $w.r.t$ x. Symbolically we write it as

$$y_1 = \frac{dy}{dx} = Dy = f'(x), \text{ where } D = \frac{d}{dx}$$

The derivative of the first derivative is called as the second derivative of y $w.r.t$ x.

ie., $$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D(Dy) = \{f'(x)\}'$$

or $$y_2 = \frac{d^2 y}{dx^2} = D^2 y = f''(x), \text{ where } D^2 = \frac{d^2}{dx^2}$$

So, in general the derivative of the $(n-1)^{th}$ derivative of y $w.r.t$ x is called as the n^{th} **derivative of** y $w.r.t$ x. Symbolically

$$y_n = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right) = D(D^{n-1} y) = \left\{f^{(n-1)}(x)\right\}'$$

Thus $y_n = \frac{d^n y}{dx^n} = D^n y = f^{(n)}(x)$ where $D^n = \frac{d^n}{dx^n}$, represents the n^{th} **derivative** of y $w.r.t$ x. $y_1, y_2 \ldots$ are respectively called as the derivatives of order one, two, ... etc. $y = f(x)$ is equivalent to $y_0 = f^{(0)}(x)$ and this can be regarded as the derivative of order zero.

If an expression represents the n^{th} derivative of a function obviously it must give first derivative, second derivative ... corresponding to $n = 1, 2, \ldots$. Now the question is, can we find the n^{th} derivative of any function? The answer is *no*. The answer for this question is *yes* if only we are able to notice a *sequential change* from one derivative to the other. Suppose we have numbers 1, 4, 9, 16, ... and try to guess the n^{th} term, it is possible if only we re-write them as $1^2, 2^2, 3^2, 4^2 \ldots$ so that the n^{th} term of the sequence is n^2. Similarly the numbers 1, 2, 6, 24, 120, ... should be put in the form 1!, 2!, 3!, 4!, 5!, ... so that the n^{th} term of the sequence is $n!$. So we should have a sequence or create a sequence (*by alternative representation*) from one derivative to the succeeding derivative for deriving the n^{th} derivative of a given function which definitely gives first, second, third, ... derivatives of the given function corresponding to $n = 1, 2, 3 \ldots$.

We shall now proceed to derive the n^{th} derivatives of some standard functions.

1.13 n^{th} derivatives of some standard functions

1. $y = e^{ax}$

$$y_1 = \frac{dy}{dx} = a\,e^{ax};\quad y_2 = \frac{d}{dx}(a\,e^{ax}) = a \cdot a\,e^{ax} = a^2 e^{ax}$$

$$y_3 = \frac{d}{dx}(a^2 e^{ax}) = a^3 e^{ax} \ldots$$

(*We should notice the sequence* $a = a^1$ *in* y_1 a^2 *in* y_2, a^3 *in* $y_3 \ldots$ *with the term* e^{ax} *fixed*)

Thus $\boxed{y_n = D^n(e^{ax}) = a^n e^{ax}}$

2. $y = a^{mx}$

We know that $\frac{d}{dx}(a^x) = a^x \log a$

$$y_1 = \frac{d}{dx}(a^{mx}) = a^{mx}\log a \cdot \frac{d}{dx}(mx) = m \log a \cdot a^{mx}$$

$$y_2 = m \log a \cdot \frac{d}{dx}(a^{mx}) = m \log a \cdot \{m \log a \cdot a^{mx}\}$$

ie., $y_2 = (m \log a)^2 a^{mx}$

$y_3 = (m \log a)^3 a^{mx}$ and so on.

Thus $\boxed{y_n = D^n(a^{mx}) = (m \log a)^n a^{mx}}$

Aliter

We know that $a = e^{\log a}$ $\therefore$ $y = a^{mx} = (e^{\log a})^{mx} = (e^{m \log a})^x$

i.e., $y = e^{bx}$ where $b = m \log a$

Hence $y_n = b^n e^{bx}$ [*by result* (1)]

Thus $$\boxed{D^n (a^{mx}) = (m \log a)^n a^{mx}}$$

3. $y = (ax + b)^m$

where m is a positive integer and $m > n$

$$y_1 = m(ax+b)^{m-1} \cdot a$$

$$y_2 = m(m-1)(ax+b)^{m-2} \cdot a^2$$

$$y_3 = m(m-1)(m-2)(ax+b)^{m-3} a^3$$

....

(It may be noticed that the factors of m is being accumulated according to the order of the derivative. Further the ending factor is $m = (m-0)$ in respect of y_1, $(m-1)$ in respect of y_2, $(m-2)$ in respect of y_3 etc. Obviously it must be $[m-(n-1)]$ in respect of y_n)

Thus $$\boxed{y_n = m(m-1)(m-2)\ldots[m-(n-1)](ax+b)^{m-n} a^n}$$

4. $y = \dfrac{1}{ax+b}$

Let us write $y = (ax+b)^{-1}$

$$y_1 = -1(ax+b)^{-2} \cdot a = (-1)^1 1!(ax+b)^{-2} \cdot a$$

$$y_2 = (-1)(-2)(ax+b)^{-3} \cdot a^2 = (-1)^2 2!(ax+b)^{-3} \cdot a^2$$

$$y_3 = (-1)(-2)(-3)(ax+b)^{-4} a^3 = (-1)^3 3!(ax+b)^{-4} a^3$$

....

$$\therefore \quad y_n = (-1)^n n!(ax+b)^{-(n+1)} a^n$$

Thus $$\boxed{y_n = D^n\left[\frac{1}{ax+b}\right] = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}}$$

5. $y = \log(ax+b)$

$$y_1 = \frac{1}{ax+b} \cdot a = a(ax+b)^{-1}$$

$$y_2 = a(-1)(ax+b)^{-2} \cdot a = a^2(-1)^1 1!(ax+b)^{-2}$$

$$y_3 = a^2(-1)(-2)(ax+b)^{-3} \cdot a = a^3(-1)^2 2!(ax+b)^{-3}$$

Similarly $y_4 = a^4(-1)^3 3!(ax+b)^{-4}$

. . . .

(It must be noticed that the expression for y_1 can be written as $a^1(-1)^0 0!(ax+b)^{-1}$, since $(-1)^0 = 1 = 0!$. With this it is convenient to notice the sequence in R.H.S corresponding to $y_1, y_2, y_3, y_4, \ldots$)

$\therefore$ $y_n = a^n(-1)^{n-1}(n-1)!(ax+b)^{-n}$

Thus $$\boxed{y_n = D^n[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}}$$

6. $y = \sin(ax+b)$

$$y_1 = \cos(ax+b) \cdot a$$

[The successive derivatives will be $-a^2\sin(ax+b)$, $-a^3\cos(ax+b)$, $a^4\sin(ax+b)$ and so on. It is not possible to create a sequence in this manner. Hence we need to put y_1 similar to y by using the trigonometric allied angle formula that $\cos\theta = \sin(\pi/2+\theta)$]

Thus $y_1 = a\sin\left[\frac{\pi}{2} + ax + b\right]$

Now $y_2 = a\cos\left[\frac{\pi}{2} + ax + b\right] \cdot a$

or $y_2 = a^2\sin\left[\frac{\pi}{2} + \left(\frac{\pi}{2} + ax + b\right)\right]$

$\therefore$ $y_2 = a^2\sin\left(2 \cdot \frac{\pi}{2} + ax + b\right)$

Similarly $y_3 = a^3\sin\left(3 \cdot \frac{\pi}{2} + ax + b\right)$

. . . .

Thus $$\boxed{y_n = D^n[\sin(ax+b)] = a^n\sin\left(n\frac{\pi}{2} + ax + b\right)}$$

7. $y = \cos(ax+b)$

$y_1 = -a\sin(ax+b)$. But $-\sin\theta = \cos(\pi/2+\theta)$

$\therefore \quad y_1 = a\cos\left(\frac{\pi}{2}+ax+b\right)$

$y_2 = -a^2\sin\left(\frac{\pi}{2}+ax+b\right)$

ie., $\quad y_2 = a^2\cos\left[\frac{\pi}{2}+\left(\frac{\pi}{2}+ax+b\right)\right]$

$\therefore \quad y_2 = a^2\cos\left(2\cdot\frac{\pi}{2}+ax+b\right)$.

Similarly $y_3 = a^3\cos\left(3\cdot\frac{\pi}{2}+ax+b\right)$

. . . .

Thus $\boxed{y_n = D^n[\cos(ax+b)] = a^n\cos\left(n\frac{\pi}{2}+ax+b\right)}$

8. $y = e^{ax}\sin(bx+c)$

$y_1 = e^{ax}\cdot b\cos(bx+c)+a\,e^{ax}\sin(bx+c)$,

by applying product rule.

ie., $\quad y_1 = e^{ax}[\,b\cos(bx+c)+a\sin(bx+c)\,]$

(The number of terms in the successive derivatives which is 2 in y_1 will become 4 in y_2, 8 in y_3 etc. if we proceed as it is. We have to plan to put y_1 similar to the form of y. This is possible by a special substitution)

Let us take the substitution $a = r\cos\theta,\ b = r\sin\theta$ as it is possible to express the newly introduced variables r and θ in terms of a and b by simple elimination.

Squaring and adding we get $a^2+b^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$

Dividing we get $b/a = \tan\theta$

Hence $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$

We shall use the substitution for the constants a and b present in the R.H.S of y_1 at only two places so that we can simplify. Thus

$$y_1 = e^{ax}[\,r\sin\theta\cos(bx+c)+r\cos\theta\sin(bx+c)\,]$$

ie., $\quad y_1 = r\,e^{ax}\sin(\theta+\overline{bx+c})$, where we have used the formula

$\sin A \cos B + \cos A \sin B = \sin(A + B)$

(It can be seen that y_1 has assumed a form similar to that of y.)

Differentiating again and simplifying as before we can obtain,

$$y_2 = r^2 e^{ax} \sin(2\theta + bx + c).$$

Similarly $\quad y_3 = r^3 e^{ax} \sin(3\theta + bx + c)$

. . . .

Thus $\quad y_n = r^n e^{ax} \sin(n\theta + bx + c)$,

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$

Thus $\boxed{D^n[e^{ax} \sin(bx + c)] = (\sqrt{a^2 + b^2})^n e^{ax} \sin[n \tan^{-1}(b/a) + bx + c]}$

9. $\quad y = e^{ax} \cos(bx + c)$

$y_1 = e^{ax} \cdot -b \sin(bx + c) + a e^{ax} \cos(bx + c)$, by product rule.

ie., $\quad y_1 = e^{ax}[a \cos(bx + c) - b \sin(bx + c)]$

Let us put $a = r\cos\theta$, and $b = r \sin\theta$.

$\therefore \quad a^2 + b^2 = r^2$ and $\tan\theta = b/a$

ie., $\quad r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$

Now, $\quad y_1 = e^{ax}[r \cos\theta \cos(bx + c) - r \sin\theta \sin(bx + c)]$

ie., $\quad y_1 = r e^{ax} \cos(\theta + bx + c)$ where we have used the formula

$\cos A \cos B - \sin A \sin B = \cos(A + B)$

Differentiating again and simplifying as before,

$$y_2 = r^2 e^{ax} \cos(2\theta + bx + c).$$

Similarly $y_3 = r^3 e^{ax} \cos(3\theta + bx + c)$.

. . . .

Thus $\quad y_n = r^n e^{ax} \cos(n\theta + bx + c)$,

where $r = \sqrt{a^2 + b^2}$ and $0 = \tan^{-1}(b/a)$.

Thus $\boxed{D^n[e^{ax} \cos(bx + c)] = (\sqrt{a^2 + b^2})^n e^{ax} \cos[n \tan^{-1}(b/a) + bx + c)}$

The formulae list of n^{th} derivatives of some standard functions we derived are presented together for ready reference.

	$y = f(x)$	$y_n = D^n y$
F_1	e^{ax}	$a^n e^{ax}$
F_2	a^{mx}	$(m \log a)^n a^{mx}$
F_3	$(ax+b)^m,\ m > n$	$m(m-1)(m-2)\ldots[m-(n-1)]\,a^n (ax+b)^{m-n}$
F_4	$\dfrac{1}{ax+b}$	$\dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$
F_5	$\log(ax+b)$	$\dfrac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$
F_6	$\sin(ax+b)$	$a^n \sin\left(\dfrac{n\pi}{2} + ax + b\right)$
F_7	$\cos(ax+b)$	$a^n \cos\left(\dfrac{n\pi}{2} + ax + b\right)$
F_8	$e^{ax}\sin(bx+c)$	$(\sqrt{a^2+b^2})^n e^{ax} \sin[\,n \tan^{-1}(b/a) + bx + c\,]$
F_9	$e^{ax}\cos(bx+c)$	$(\sqrt{a^2+b^2})^n e^{ax} \cos[\,n \tan^{-1}(b/a) + bx + c\,]$

Remark.

Observe similarities in the pair of formulae F_4 & F_5 ; F_6 & F_7 ; F_8 & F_9 as it would help to remember the formulae easily.

Illustrative Examples

Here are a few examples for getting an acquaintance with the formulae of n^{th} derivative of standard functions.

1. $y = e^{3x}$ $\quad\therefore\quad y_n = 3^n e^{3x} \ldots \quad (a = 3 \text{ in } F_1)$

2. $y = a^{3x}$ $\quad\therefore\quad y_n = (3 \log a)^n a^{3x} \ldots (m = 3, \text{ in } F_2)$

3. $y = 3^{5x}$ $\quad\therefore\quad y_n = (5 \log 3)^n 3^{5x} \ldots (a = 3,\ m = 5 \text{ in } F_2)$

4. $y = \dfrac{1}{3x+2}$ $\quad\therefore\quad y_n = \dfrac{(-1)^n n!\, 3^n}{(3x+2)^{n+1}} \ldots (a = 3,\ b = 2 \text{ in } F_4)$

5. $y = \log(2x+5)$ $\quad \therefore\ y_n = \dfrac{(-1)^{n-1}(n-1)!\,2^n}{(2x+5)^n}$ ...($a = 2,\ b = 5$ in F_5)

6. $y = \dfrac{1}{(x+1)^2}$ or $y = (x+1)^{-2}$ ($m = -2,\ a = 1,\ b = 1$ in F_3)

$\therefore$ $y_n = \{(-2)(-3)(-4)\ldots(-2-(n-1))\}(x+1)^{-2-n}\cdot 1^n$

ie., $y_n = (-1)^n \cdot 2.\ 3.\ 4\ldots(n+1)(x+1)^{-(n+2)}$. Here $1^n = 1$

(Observe that -1 is a common factor in all the n terms)

$$\therefore\ y_n = \frac{(-1)^n(n+1)!}{(x+1)^{n+2}}$$

7. $y = \cos(4x+3)$ $\therefore\ y_n = 4^n \cos\left(\dfrac{n\pi}{2} + 4x + 3\right)$ ... ($a = 4,\ b = 3$ in F_7)

8. $y = \sin 6x$ $\therefore\ y_n = 6^n \sin\left(\dfrac{n\pi}{2} + 6x\right)$ ... ($a = 6,\ b = 0$ in F_6)

9. $y = e^{2x}\cos 3x$ ... ($a = 2,\ b = 3,\ c = 0$ in F_9)

$\therefore$ $y_n = (\sqrt{13})^n e^{2x}\cos[n\tan^{-1}(3/2) + 3x]$

10. $y = e^x \sin x$ ... ($a = 1,\ b = 1,\ c = 0$ in F_8)

$\therefore$ $y_n = (\sqrt{2})^n e^x \sin(n\tan^{-1}1 + x) = (\sqrt{2})^n e^x \sin\left(\dfrac{n\pi}{4} + x\right)$

11. $y = \log_{10}(4x^2 - 1)$

>> *It is importnt to note that we have to convert the logarithm to the base e by the property.*

$$\log_{10} x = \frac{\log_e x}{\log_e 10}$$

$$\therefore\ y = \frac{\log_e(4x^2-1)}{\log_e 10} = \frac{\log_e(2x-1) + \log_e(2x+1)}{\log_e 10}$$

Thus $y_n = \dfrac{1}{\log_e 10}\left\{\dfrac{(-1)^{n-1}(n-1)!\,2^n}{(2x-1)^n} + \dfrac{(-1)^{n-1}(n-1)!\,2^n}{(2x+1)^n}\right\}$

12. $y = \cos h^2 3x$

>> $y = \left[\dfrac{e^{3x} + e^{-3x}}{2} \right]^2 = \dfrac{1}{4} \left[e^{6x} + e^{-6x} + 2 \right]$

Thus $y_n = \dfrac{1}{4} \left[6^n e^{6x} + (-6)^n e^{-6x} \right]$

13. $y = e^{2x} \cos h\, 4x$

>> $y = e^{2x} \cdot \dfrac{1}{2} \left(e^{4x} + e^{-4x} \right) = \dfrac{1}{2} \left(e^{6x} + e^{-2x} \right)$

Thus $y_n = \dfrac{1}{2} \left\{ 6^n e^{6x} + (-2)^n e^{-2x} \right\}$

14. $y = \cos^2 x$

>> $y = \dfrac{1}{2} (1 + \cos 2x)$

Thus $y_n = \dfrac{1}{2} \left\{ 0 + 2^n \cos \left(\dfrac{n\pi}{2} + 2x \right) \right\} = 2^{n-1} \cos \left(\dfrac{n\pi}{2} + 2x \right)$

15. $y = \sin 8x \cdot \cos 5x$

>> $y = \dfrac{1}{2} \left\{ \sin (8x + 5x) + \sin (8x - 5x) \right\} = \dfrac{1}{2} (\sin 13x + \sin 3x)$

Thus, $y_n = \dfrac{1}{2} \left\{ (13)^n \sin \left(\dfrac{n\pi}{2} + 13x \right) + 3^n \sin \left(\dfrac{n\pi}{2} + 3x \right) \right\}$

1.14 Leibnitz theorem for the n^{th} derivative of a product

Statement : *If u and v are functions of x then*

$$D^n (uv) = (uv)_n = uv_n + {}^nC_1\, u_1 v_{n-1} + {}^nC_2\, u_2 v_{n-2} + \ldots + u_n v$$

Working procedure for problems

- In problems, to find the n^{th} derivative of a given product involving polynomial functions like x, x^2, $(x+1)^3$ etc. we have to take them as the first function 'u' since $D^n(u) = u_n$ will become zero for some n, with the result the process terminates at some stage.
- In some problems we have to use our discretion for getting the result in some specific form.

WORKED PROBLEMS

Find the n^{th} derivative of the following functions.

1. $x^3 4^x$	2. $x^2 \cos^2 3x$	3. $x^2 \log 4x$
4. $x e^{2x} \sin 3x \cos 2x$	5. $e^{-x} x^2 \sinh 2x$	6. $e^x \log x$

1. $y = x^3 4^x$

Let $u = x^3,\ v = 4^x$

We have Leibnitz theorem,

$$(uv)_n = uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + \ldots + u_n v$$

But $${}^nC_1 = n,\ {}^nC_2 = \frac{n(n-1)}{1 \cdot 2},\ {}^nC_3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$$

We also have $v_n = D^n(4^x) = (\log 4)^n 4^x$ and $u_1 = D(x^3) = 3x^2$;

$$u_2 = D(3x^2) = 6x\ ;\ u_3 = D(6x) = 6,\ u_4 = D(6) = 0$$

Now applying Leibnitz theorem we have,

$$y_n = x^3 \cdot (\log 4)^n 4^x + n \cdot 3x^2 \cdot (\log 4)^{n-1} 4^x + \frac{n(n-1)}{2} \cdot 6x \cdot (\log 4)^{n-2} \cdot 4^x + \frac{n(n-1)(n-2)}{6} \cdot 6 \cdot (\log 4)^{n-3} \cdot 4^x$$

Thus $$\mathbf{y_n = (\log 4)^{n-3} \cdot 4^x \{x^3(\log 4)^3 + 3n x^2 (\log 4)^2 + 3n(n-1)x(\log 4) + n(n-1)(n-2)\}}$$

2.. $y = x^2 \cos^2 3x$

$$= x^2\left(\frac{1+\cos 6x}{2}\right) = \frac{x^2}{2} + \frac{1}{2}(x^2 \cos 6x)$$

$$\therefore \quad y_n = D^n\left(\frac{x^2}{2}\right) + \frac{1}{2} D^n (x^2 \cos 6x)$$

ie., $$y_n = 0 + \frac{1}{2} D^n (x^2 \cos 6x) \quad \text{where } n > 2$$

Now, let $u = x^2$ and $v = \cos 6x$.

Applying Leibnitz theorem we get

$$y_n = \frac{1}{2}\left\{x^2 (\cos 6x)_n + n \cdot 2x \cdot (\cos 6x)_{n-1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2(\cos 6x)_{n-2}\right\}$$

We have $(\cos 6x)_n = D^n(\cos 6x) = 6^n \cos\left(\frac{n\pi}{2} + 6x\right)$

$$\therefore \quad y_n = \frac{1}{2}\left\{x^2 \quad 6^n \cos\left(\frac{n\pi}{2} + 6x\right) + 2nx \cdot 6^{n-1} \cos\left((n-1)\frac{\pi}{2} + 6x\right)\right.$$

$$\left. + n(n-1) \cdot 6^{n-2} \cos\left((n-2)\frac{\pi}{2} + 6x\right)\right\}$$

Thus $$y_n = \frac{6^{n-2}}{2}\left\{36x^2 \cos\left(\frac{n\pi}{2} + 6x\right)\right.$$

$$\left. + 12nx \cos\left((n-1)\frac{\pi}{2} + 6x\right) + n(n-1)\cos\left((n-2)\frac{\pi}{2} + 6x\right)\right\}$$

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3. $y = x^2 \log 4x$

Let $u = x^2, \ v = \log 4x$

Applying Leibnitz theorem we have,

$$y_n = x^2 (\log 4x)_n + n \cdot 2x \cdot (\log 4x)_{n-1} + \frac{n(n-1)}{2} \cdot 2(\log 4x)_{n-2}$$

We have $(\log 4x)_n = D^n(\log 4x)$

$$= \frac{(-1)^{n-1}(n-1)!\,4^n}{(4x)^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

Now $$y_n = x^2 \cdot \frac{(-1)^{n-1}(n-1)!}{x^n} + 2nx \cdot \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$$

$$+ n(n-1)\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}$$

Thus $$y_n = \frac{1}{x^{n-2}}\left\{(-1)^{n-1}(n-1)! + 2n(-1)^{n-2}(n-2)!\right.$$

$$\left. + n(n-1)(-1)^{n-3}(n-3)!\right\} \quad \text{where } n > 2$$

Since $(n-3)!$ is valid only when $n > 2$, we claim that the above expression for y_n is valid for $n > 2$

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4. $y = x e^{2x} \sin 3x \cos 2x$

$$y = x e^{2x} \cdot \frac{1}{2}\left[\sin 5x + \sin x \right]$$

$$\therefore \quad y_n = \frac{1}{2}\left[D^n\left(x \cdot e^{2x} \sin 5x\right) + D^n\left(x \cdot e^{2x} \sin x\right) \right] \qquad \ldots (1)$$

We have to apply Leibnitz theorem separately for the two terms by taking $u = x$, $v = e^{2x} \sin 5x$ in the first term and $u = x$, $v = e^{2x} \sin x$ in the second term.
Further we have,

$$v_n = (\sqrt{29})^n e^{2x} \sin\left(n \tan^{-1}(5/2) + 5x\right)$$ in the first term and

$$v_n = (\sqrt{5})^n e^{2x} \sin\left(n \tan^{-1}(1/2) + x\right)$$ in the second term.

Thus $$y_n = \frac{1}{2}\left[x \cdot (\sqrt{29})^n e^{2x} \sin\left\{n \tan^{-1}(5/2) + 5x\right\} + n \cdot 1 \cdot (\sqrt{29})^{n-1} e^{2x} \sin\left\{(n-1)\tan^{-1}(5/2) + 5x\right\}\right]$$
$$+ \frac{1}{2}\left[x \cdot (\sqrt{5})^n e^{2x} \sin\left\{n \tan^{-1}(1/2) + x\right\} + n \cdot 1 \cdot (\sqrt{5})^{n-1} e^{2x} \sin\left\{(n-1)\tan^{-1}(1/2) + x\right\}\right]$$

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5. $y = e^{-x} x^2 \sinh 2x$

$$y = e^{-x} \cdot x^2 \cdot \frac{1}{2}(e^{2x} - e^{-2x})$$

ie., $$y = \frac{1}{2}\left(x^2 e^x - x^2 e^{-3x}\right)$$

$$\therefore \quad y_n = \frac{1}{2}\left[D^n\left(x^2 e^x\right) - D^n\left(x^2 e^{-3x}\right) \right]$$

Applying Leibnitz theorem to each of the two terms we get,

$$y_n = \frac{1}{2}\left[x^2 e^x + n \cdot 2x \cdot e^x + \frac{n(n-1)}{2} \cdot 2 \cdot e^x \right]$$
$$- \frac{1}{2}\left[x^2 (-3)^n e^{-3x} + n \cdot 2x \cdot (-3)^{n-1} e^{-3x} + \frac{n(n-1)}{2} \cdot 2(-3)^{n-2} e^{-3x}\right.$$

ie., $$y_n = \frac{e^x}{2}\left[x^2 + 2nx + n(n-1) \right] - (-3)^{n-2} \frac{e^{-3x}}{2}\left[9x^2 - 6nx + n(n-1) \right]$$

---

6. $y = e^x \log x$

Here neither of the functions is a polynomial and either of them could be the first function.

Let us take $u = e^x$ and $v = \log x$.

Applying Leibnitz theorem we have,

$$y_n = e^x D^n(\log x) + n\,e^x D^{n-1}(\log x) + \frac{n(n-1)}{2!} e^x D^{n-2}(\log x) + \cdots + D^n(e^x) \cdot \log x$$

$$\therefore \quad y_n = e^x \cdot \frac{(-1)^{n-1}(n-1)!}{x^n} + n\,e^x \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + \frac{n(n-1)}{2} e^x \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} + \cdots + e^x \log x$$

$$y_n = e^x \left\{ \frac{(-1)^{n-1}(n-1)!}{x^n} + \frac{n(-1)^{n-2}(n-2)!}{x^{n-1}} + \frac{n(n-1)}{2} \cdot \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} + \cdots + \log x \right\}$$

---

7. *Show that*

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n n!}{x^{n+1}} \left\{ \log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} \right\}$$

>> Let $y = \dfrac{\log x}{x} = \log x \cdot \dfrac{1}{x}$ and let $u = \log x$, $v = \dfrac{1}{x}$

We have Leibnitz theorem,

$$(uv)_n = uv_n + {}^nC_1\, u_1 v_{n-1} + {}^nC_2\, u_2 v_{n-2} + \cdots + u_n v \qquad \ldots (1)$$

Now, $u = \log x \quad \therefore \quad u_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

$v = \dfrac{1}{x} \quad \therefore \quad v_n = \dfrac{(-1)^n n!}{x^{n+1}}$

Using these in (1) by taking appropriate values for n we get,

$$D^n\left(\frac{\log x}{x}\right) = \log x \cdot \frac{(-1)^n n!}{x^{n+1}} + n\,\frac{1}{x} \cdot \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$+ \frac{n(n-1)}{1 \cdot 2} \cdot \left(-\frac{1}{x^2}\right)\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + \dots + \frac{(-1)^{n-1}(n-1)!}{x^n} \cdot \frac{1}{x}$$

ie.,
$$= \log x \cdot \frac{(-1)^n n!}{x^{n+1}} + \frac{(-1)^{n-1} n!}{x^{n+1}} - \frac{(-1)^{n-2} n!}{2x^{n+1}} + \dots + \frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x + (-1)^{-1} - \frac{(-1)^{-2}}{2} + \dots + \frac{(-1)^{-1}(n-1)!}{n!}\right]$$

Note : $(-1)^{-1} = \frac{1}{-1} = -1$; $(-1)^{-2} = \frac{1}{(-1)^2} = 1$

Also $\frac{(n-1)!}{n!} = \frac{(n-1)!}{n \cdot (n-1)!} = \frac{1}{n}$

Thus $\frac{d^n}{dx^n}\left[\frac{\log x}{x}\right] = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} \dots - \frac{1}{n}\right]$

---

8. *Prove that*

$$D^n\,[x^n \log x] = n!\left\{\log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right\}$$

>> Let $y = x^n \log x$.

It is important to take a note that the given function is involved with n and hence the n^{th} derivative $y_n = D^n(x^n \log x)$ is to be viewed as follows.

$y_1 = D^1(x \log x)$; $y_2 = D^2(x^2 \log x)$; $y_3 = D^3(x^3 \log x)$ etc.,

$n = 1, 2, 3 \dots$ is to be correlated both in L.H.S and R.H.S.

Since we have to prove a result for positive integral values of n, the method of mathematical induction is well suited.

Step-1 : We shall verify the result when $n = 1$.

L.H.S $= D^1(x \log x) = \frac{d}{dx}(x \log x) = x \cdot \frac{1}{x} + \log x = 1 + \log x$

R.H.S $= 1!(\log x + 1) = \log x + 1$

$\therefore$ L.H.S = R.H.S $\Rightarrow$ The result is true when $n = 1$

Step-2 : We shall assume the result to be true for $n = k$, where k is any positive integer.

$$\therefore \quad D^k\left(x^k \log x\right) = k!\left\{\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right\} \qquad \ldots (1)$$

Step-3 : We have to prove the result for $n = k+1$.

$$D^{k+1}\left(x^{k+1} \log x\right) = D^k\left[D\left(x^{k+1} \log x\right)\right]$$

$$= D^k\left[x^{k+1} \cdot \frac{1}{x} + (k+1)\, x^k \cdot \log x\right]$$

ie., $$D^{k+1}\left(x^{k+1} \log x\right) = D^k\left[x^k\right] + (k+1)\, D^k\left[x^k \log x\right] \qquad \ldots (2)$$

Let us consider the first term $D^k\left[x^k\right]$

$$D(x^k) = k x^{k-1},\ D^2(x^k) = k(k-1)\, x^{k-2},$$

$$D^3(x^k) = k(k-1)(k-2)\, x^{k-3} \text{ etc.}$$

$$D^k(x^k) = k(k-1)(k-2)\cdots 3 \cdot 2 \cdot 1\, x^{k-k} = k! \cdot 1 = k!$$

$$\therefore \quad D^k(x^k) = k!$$

Note : Remember $D^n(x^n) = n!$

Using this result and (1) in the R.H.S of (2) we have

$$D^{k+1}\left(x^{k+1} \log x\right) = k! + (k+1)\left\{k!\left(\log x + 1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\right\}$$

ie., $$= k! + (k+1)!\left\{\log x + 1 + \frac{1}{2} + \cdots + \frac{1}{k}\right\}$$

$$= (k+1)!\left\{\log x + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{k!}{(k+1)!}\right\}$$

$$D^{k+1}\left(x^{k+1} \log x\right) = (k+1)!\left\{\log x + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1}\right\} \qquad \ldots (3)$$

Comparing (1) and (3) we conclude that the result is true for $n = k+1$.

Hence by the principle of mathematical induction the result is true for any positive integer n.

Thus $$D^n\left[x^n \log x\right] = n!\left\{\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right\}$$

---

9. If $y_n = D^n\left(x^n \log x\right)$ prove that $y_n = n\, y_{n-1} + (n-1)!$ and hence deduce that

$$y_n = n!\left\{\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right\}$$

Remark : *The function given in this example is same as the previous example. But we have to arrive at the result as directed in the problem.*

>> $y_n = D^n(x^n \log x) = D^{n-1}\{D(x^n \log x)\}$

$$= D^{n-1}\left\{x^n \cdot \frac{1}{x} + n\,x^{n-1}\log x\right\}$$

$$= D^{n-1}\left(x^{n-1}\right) + n\,D^{n-1}\left(x^{n-1}\log x\right)$$

$\therefore$ $\mathbf{y_n = (n-1)! + n\,y_{n-1}}$. This proves the first part.

Now Putting the values for $n = 1, 2, 3$...we get

$$y_1 = 0! + 1 \cdot y_0 = 1 + \log x = 1!(\log x + 1)$$

$$y_2 = 1! + 2y_1 = 1 + 2(1 + \log x)$$

ie., $y_2 = 2\log x + 3 = 2\left(\log x + 3/2\right) = 2!\left(\log x + 1 + \frac{1}{2}\right)$

$$y_3 = 2! + 3y_2 = 2 + 3(2\log x + 3)$$

ie., $y_3 = 6\log x + 11 = 6\left(\log x + 11/6\right) = 3!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3}\right)$

. . . .

Thus $\mathbf{y_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)}$.

---

10. *If* $y = x^n \log x$, *show that* $y_{n+1} = \dfrac{n!}{x}$

>> Observing that the desired result is related to y_{n+1} being the $(n+1)^{th}$ derivative, it will be convenient to first obtain y_1 and then differentiate n times.

Consider $y = x^n \log x$ $\therefore$ $y_1 = x^n \cdot \frac{1}{x} + n\,x^{n-1}\log x = x^{n-1} + n\,x^{n-1}\log x$

Multiplying by x we get,

$$x\,y_1 = x^n + n\,x^n \log x = x^n + ny$$

Now, $D^n(x\,y_1) = D^n(x^n) + n\,D^n(y)$

Applying Leibnitz theorem to L.H.S and using $D^n(x^n) = n!$ we have,

$$x\,D^n(y_1) + {}^nC_1 \cdot 1 \cdot D^{n-1}(y_1) = n! + n\,y_n$$

ie., $x\,y_{n+1} + n\,y_n = n! + n\,y_n$

$\Rightarrow$ $x\,y_{n+1} = n!$ or $y_{n+1} = \dfrac{n!}{x}$

Note : *The function involved in Examples - 8, 9, 10 is the same and different technique is adopted in each problem to get the result as desired.*

---

11. *If* $y = \tan x$, *prove that*
$$y_n(0) - {}^nC_2\, y_{n-2}(0) + {}^nC_4\, y_{n-4}(0) - \cdots = \sin(n\pi/2)$$

>> $y = \tan x = \dfrac{\sin x}{\cos x}$

$\therefore$ $\cos x \cdot y = \sin x$ and differentiating n times we have,

$D^n(\cos x \cdot y) = D^n(\sin x)$

Applying Leibnitz theorem onto the L.H.S and making use of the standard formula in R.H.S we get,

$$\cos x \cdot y_n + {}^nC_1(-\sin x)\cdot y_{n-1} + {}^nC_2(-\cos x)\cdot y_{n-2}$$
$$+ {}^nC_3(\sin x)\cdot y_{n-3} + {}^nC_4(\cos x)\cdot y_{n-4} + \cdots = \sin(n\pi/2 + x) \quad \ldots (1)$$

Here $y_n, y_{n-1}, \ldots$ are all functions of $x : y_n(x), y_{n-1}(x) \ldots$ and $y_n(0)$ is to be understood as the value of y_n when $x = 0$. In this context, observing the desired result we need to put $x = 0$ in (1). $\sin 0 = 0$, $\cos 0 = 1$

Thus (1) now becomes,

$$\boldsymbol{y_n(0) - {}^nC_2\, y_{n-2}(0) + {}^nC_4\, y_{n-4}(0) - \cdots = \sin(n\pi/2)}$$

Standard type of problems on Leibnitz theorem

Working procedure for problems

- Given y as a function of x (*explicit or implicit*) we need to establish a relation involving y_{n+2}, y_{n+1} and y_n or y_{n+1}, y_n and y_{n-1} etc.
- In order to establish such a relation we have to first establish a relation involving y_2, y_1, y or y_1, y as the case may be by differentiating and simplifying judiciously.
- Later we have to differentiate the relation so obtained n times and Leibnitz theorem has to be employed in differentiating a product involved in the relation.

Observe the following :

$$D^n(y_2) = y_{n+2},\ D^{n-1}(y_2) = y_{n+1},\ D^{n-2}(y_2) = y_n,$$
$$D^{n+1}(y_1) = y_{n+2},\ D^{n+1}(y) = y_{n+1}\ \textit{etc.}$$

WORKED PROBLEMS

12. *If* $y = a\cos(\log x) + b\sin(\log x)$ *show that* $x^2 y_2 + x y_1 + y = 0$.
Then apply Leibnitz theorem to differentiate this result n times.

or

If $y = a\cos(\log x) + b\sin(\log x)$
show that $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$.

\>\> $y = a\cos(\log x) + b\sin(\log x)$

Differentiate *w.r.t* x

$\therefore \quad y_1 = -a\sin(\log x)\cdot\dfrac{1}{x} + b\cdot\cos(\log x)\cdot\dfrac{1}{x}$ *(we avoid quotient rule to find* y_2*)*

$\Rightarrow \quad x y_1 = -a\sin(\log x) + b\cos(\log x)$

Differentiating again *w.r.t* x we have,

$$x y_2 + 1\cdot y_1 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

or $\quad x^2 y_2 + x y_1 = -\left[a\cos(\log x) + b\sin(\log x)\right] = -y$

Thus $\quad \mathbf{x^2 y_2 + x y_1 + y = 0}$

Now we have to differentiate this result n times.

ie., $\quad D^n(x^2 y_2) + D^n(x y_1) + D^n(y) = 0$

We have to employ Leibnitz theoreom for the first two terms.
Hence we have,

$$\left\{x^2\cdot D^n(y_2) + n\cdot 2x\cdot D^{n-1}(y_2) + \frac{n(n-1)}{1\cdot 2}\cdot 2\cdot D^{n-2}(y_2)\right\} + \left\{x\cdot D^n(y_1) + n\cdot 1\cdot D^{n-1}(y_1)\right\} + y_n = 0$$

ie., $\quad \left\{x^2 y_{n+2} + 2nx y_{n+1} + n(n-1) y_n\right\} + \left\{x y_{n+1} + n y_n\right\} + y_n = 0$

ie., $\quad x^2 y_{n+2} + 2nx y_{n+1} + n^2 y_n - n y_n + x y_{n+1} + n y_n + y_n = 0$

Thus $\quad \mathbf{x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0}$

13. *If* $\tan y = x$, *prove that* $(1+x^2)y_2 + 2xy_1 = 0$ *and hence show that,*

$$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

$$\tan y = x \Rightarrow y = \tan^{-1} x$$

$$\therefore \quad y_1 = \frac{1}{1+x^2} \quad \text{or} \quad (1+x^2)y_1 = 1$$

Differentiating again *w.r.t* x we have

$$\mathbf{(1+x^2)y_2 + 2xy_1 = 0}$$

Now, differentiating this result n times we have,

$$D^n\left[(1+x^2)y_2\right] + 2D^n[xy_1] = 0$$

Applying Leibnitz theorem to each of the terms we have,

$$\left\{(1+x^2)D^n(y_2) + n \cdot 2x \cdot D^{n-1}(y_2) + \frac{n(n-1)}{1\cdot 2} \cdot 2 \cdot D^{n-2}(y_2)\right\}$$
$$+2\left\{x \cdot D^n(y_1) + n \cdot 1 \cdot D^{n-1}(y_1)\right\} = 0$$

ie., $\left\{(1+x^2)y_{n+2} + n \cdot 2x \cdot y_{n+1} + (n^2-n)y_n\right\} + 2\left\{xy_{n+1} + ny_n\right\} = 0$

ie., $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + (n^2+n)y_n = 0$

Thus $\mathbf{(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0}$

Note : *The result of this example is related to the function* $y = \tan^{-1} x$ *which can be given in different versions as follows.*

(i) *If* $y = \tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ *then show that*

$$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0.$$

>> Put $x = \tan\theta$. Then $y = \tan^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$

ie., $y = \tan^{-1}\tan(\pi/4 + \theta) = (\pi/4) + \theta$ or $y = (\pi/4) + \tan^{-1} x$

This gives $y_1 = 0 + \dfrac{1}{1+x^2}$ or $(1+x^2)y_1 = 1$

Then the result follows as in Problem - 13

(ii) *If* $y = \tan^{-1}\left(\dfrac{a+x}{a-x}\right)$, *then show that*

$$(a^2+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

>> $y = \tan^{-1}\left(\dfrac{a+x}{a-x}\right)$.

Put $x = a\tan\theta$

$\therefore$ $y = \tan^{-1}\left[\dfrac{a(1+\tan\theta)}{a(1-\tan\theta)}\right] = \tan^{-1}\tan(\pi/4+\theta) = (\pi/4)+\theta$

Thus $y = (\pi/4)+\theta$ where $x/a = \tan\theta$ or $\theta = \tan^{-1}(x/a)$

Now $y = (\pi/4)+\tan^{-1}(x/a)$

$\therefore$ $y_1 = 0 + \dfrac{1}{1+\dfrac{x^2}{a^2}}\cdot\dfrac{1}{a} = \dfrac{a^2}{a^2+x^2}\cdot\dfrac{1}{a} = \dfrac{a}{a^2+x^2}$ or $(a^2+x^2)\,y_1 = a.$

Differentiating *w.r.t* x we get,

$$(a^2+x^2)\,y_2 + 2x\,y_1 = 0 \qquad \ldots(1)$$

This part of the result is similar to the result in the first part of Problem - 13. as we have a^2 instead of 1. Proceeding on the same lines in applying Leibnitz theorem to both the terms of (1) we can can obtain,

$$\boldsymbol{(a^2+x^2)\,y_{n+2} + 2(n+1)\,x\,y_{n+1} + n(n+1)\,y_n = 0}$$

---

14. *If* $\cos^{-1}(y/b) = \log(x/n)^n$, *then show that*

$$x^2 y_{n+2} + (2n+1)\,x\,y_{n+1} + 2n^2 y_n = 0$$

>> By data, $\cos^{-1}(y/b) = n\log(x/n)$ $\because$ $\log(a^m) = m\log a$

$\Rightarrow$ $\dfrac{y}{b} = \cos\left[n\log(x/n)\right]$ or $y = b\cdot\cos\left[n\log(x/n)\right]$ $\ldots(1)$

Differentiating *w.r.t* x we get,

$$y_1 = -b\sin\left[n\log(x/n)\right]\cdot n\cdot\frac{1}{(x/n)}\cdot\frac{1}{n} \quad x\,y_1 = -n\,b\sin\left[n\log(x/n)\right]$$

Differentiating *w.r.t* x again we get,

$$x\,y_2 + 1\cdot y_1 = -n\cdot b\cos\left[n\log(x/n)\right]\cdot n\cdot\frac{1}{(x/n)}\cdot\frac{1}{n}$$

or $x(x\,y_2 + y_1) = -n^2 b\cos\left[n\log(x/n)\right] = -n^2 y$, by using (1).

or $x^2 y_2 + x\,y_1 + n^2 y = 0$

Differentiating each term n times we have,

$$D^n(x^2 y_2) + D^n(x y_1) + n^2 D^n(y) = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{x^2 y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n\right\} + \left\{x y_{n+1} + n \cdot 1 \cdot y_n\right\} + n^2 y_n = 0$$

ie., $x^2 y_{n+2} + 2nx y_{n+1} + n^2 y_n - n y_n + x y_{n+1} + n y_n + n^2 y_n = 0$

Thus $\mathbf{x^2 y_{n+2} + (2n+1) x y_{n+1} + 2n^2 y_n = 0}$

---

15. *If* $y = \dfrac{\sinh^{-1} x}{\sqrt{1+x^2}}$, *prove that*

$$(1+x^2) y_{n+2} + (2n+3) x y_{n+1} + (n+1)^2 y_n = 0$$

>> By data $y = \dfrac{\sinh^{-1} x}{\sqrt{1+x^2}}$

or $\sqrt{1+x^2}\, y = \sinh^{-1} x$ (*we always avoid denominator for convenient differentiation*)

Now, differentiating *w.r.t x* we get,

$$\sqrt{1+x^2} \cdot y_1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x \cdot y = \frac{1}{\sqrt{1+x^2}} \quad \text{or} \quad (1+x^2) y_1 + x y = 1$$

Differentiating *w.r.t x* again we get,

$$\left\{(1+x^2) y_2 + 2x y_1\right\} + \left\{x y_1 + 1 \cdot y\right\} = 0$$

ie., $(1+x^2) y_2 + 3x y_1 + y = 0$

Differentiating each term n times we have

$$D^n\left[(1+x^2) y_2\right] + 3D^n\left[x y_1\right] + D^n(y) = 0$$

Applying Leibnitz theorem to the product terms we get

$$\left\{(1+x^2) y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n\right\} + 3\left\{x y_{n+1} + n \cdot 1 \cdot y_n\right\} + y_n = 0$$

ie., $(1+x^2) y_{n+2} + 2nx y_{n+1} + n^2 y_n - n y_n + 3x y_{n+1} + 3n y_n + y_n = 0$

ie., $(1+x^2) y_{n+2} + (2n+3) x y_{n+1} + (n^2 + 2n + 1) y_n = 0$

Thus $\mathbf{(1+x^2) y_{n+2} + (2n+3) x y_{n+1} + (n+1)^2 y_n = 0}$

---

16. *If* $y = \sin \log (x^2 + 2x + 1)$

or

If $\sin^{-1} y = 2 \log (x + 1)$ *show that*

$$(x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2+4) y_n = 0$$

>> **By data** $y = \sin \log (x^2 + 2x + 1)$

$\therefore \quad y_1 = \cos \log (x^2 + 2x + 1) \cdot \dfrac{1}{x^2 + 2x + 1} \cdot 2x + 2$

ie., $\quad y_1 = \cos \log (x^2 + 2x + 1) \cdot \dfrac{1}{(x+1)^2} \, 2(x+1)$

ie., $\quad y_1 = \dfrac{2 \cos \log (x^2 + 2x + 1)}{(x+1)}$ or $(x+1) y_1 = 2 \cos \log (x^2 + 2x + 1)$

Differentiating *w.r.t* x again we get

$$(x+1) y_2 + 1 \cdot y_1 = -2 \sin \log (x^2 + 2x + 1) \frac{1}{(x+1)^2} \cdot 2(x+1)$$

or $\quad (x+1)^2 y_2 + (x+1) y_1 = -4y \quad$ or $\quad (x+1)^2 y_2 + (x+1) y_1 + 4y = 0$

Differentiating each term n times we have,

$$D^n \left[(x+1)^2 y_2 \right] + D^n \left[(x+1) y_1 \right] + 4 D^n [y] = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{ (x+1)^2 y_{n+2} + n \cdot 2(x+1) \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n \right\}$$
$$+ \left\{ (x+1) y_{n+1} + n \cdot 1 \cdot y_n \right\} + 4y_n = 0$$

ie., $\quad (x+1)^2 y_{n+2} + 2n(x+1) y_{n+1} + n^2 y_n - n y_n + (x+1) y_{n+1} + n y_n + 4y_n = 0$

Thus $\quad \boldsymbol{(x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2+4) y_n = 0}$

---

17. *If* $y = (x^2 - 1)^n$ *show that,*

$$(1 - x^2) y_{n+2} - 2x y_{n+1} + n(n+1) y_n = 0$$

>> $\quad y = (x^2 - 1)^n$

Taking logarithms on both sides, . . . (*for convenience*)

$$\log y = n \log (x^2 - 1)$$

Differentiating *w.r.t* x we get

$$\frac{1}{y}\, y_1 = n \cdot \frac{1}{x^2-1} \cdot 2x \quad \text{or} \quad (x^2-1)\, y_1 = 2\, n\, x\, y$$

Differentiating again w.r.t x we get,

$$(x^2-1)\, y_2 + 2\, x\, y_1 = 2\, n\, (x\, y_1 + y)$$

Now differentiating each term n times we have,

$$D^n\left[\,(x^2-1)\, y_2\,\right] + 2D^n\,[\,x\, y_1\,] = 2\, n\, D^n\,[\,x\, y_1\,] + 2\, n\, D^n\,[\,y\,]$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{(x^2-1)\, y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n\,(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n\right\}$$

$$+ 2\left\{x\, y_{n+1} + n \cdot 1 \cdot y_n\right\} = 2\, n\left\{x\, y_{n+1} + n \cdot 1 \cdot y_n\right\} + 2\, n\, y_n$$

ie., $$(x^2-1)\, y_{n+2} + 2\, n\, x\, y_{n+1} + n^2\, y_n - n y_n + 2\, x\, y_{n+1} + 2\, n\, y_n$$

$$= 2\, n\, x\, y_{n+1} + 2\, n^2\, y_n + 2\, n\, y_n$$

ie., $$(x^2-1)\, y_{n+2} + 2\, x\, y_{n+1} - n^2\, y_n - n y_n = 0$$

ie., $$(x^2-1)\, y_{n+2} + 2\, x\, y_{n+1} - n\,(n+1)\, y_n = 0$$

or $$\mathbf{(1-x^2)\, y_{n+2} - 2\, x\, y_{n+1} + n\,(n+1)\, y_n = 0}$$

Note : ***Alternative version of the problem :***

If $y = (x^2-1)^n$ *show that* $y_n = \dfrac{d^n y}{dx^n}$ *satisfies the equation*

$$(1-x^2)\,\frac{d^2 y}{dx^2} - 2\,x\,\frac{dy}{dx} + n\,(n+1)\,y = 0$$

>> We need to show that,

$$(1-x^2)\,\frac{d^2}{dx^2}\,(y_n) - 2\,x\,\frac{d}{dx}\,(y_n) + n\,(n+1)\,y_n = 0$$

That is to show that, $(1-x^2)\, y_n'' - 2\, x\, y_n' + n\,(n+1)\, y_n = 0$
and this is equivalent to

$$(1-x^2)\, y_{n+2} - 2\, x\, y_{n+1} + n\,(n+1)\, y_n = 0$$

This is the same result established in Problem - 17.

- -

18. *If* $y = \log(x+\sqrt{1+x^2})$, *prove that* $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n = 0$

>> By data, $y = \log(x+\sqrt{1+x^2})$

$$\therefore \quad y_1 = \frac{1}{(x+\sqrt{1+x^2})}\left\{1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right\}$$

ie., $$y_1 = \frac{1}{(x+\sqrt{1+x^2})}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} \qquad \text{or} \qquad \sqrt{1+x^2}\,y_1 = 1$$

Differentiating *w.r.t* x again we get

$$\sqrt{1+x^2}\,y_2+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\cdot y_1 = 0 \qquad \text{or} \qquad (1+x^2)y_2+xy_1 = 0$$

Now, $D^n\left[(1+x^2)y_2\right]+D^n\left[xy_1\right] = 0$

Applying Leibnitz theorem to each term we get,

$$\left\{(1+x^2)y_{n+2}+n\cdot 2x\cdot y_{n+1}+\frac{n(n-1)}{1\cdot 2}\cdot 2\cdot y_n\right\}$$
$$+\left[x\cdot y_{n+1}+n\cdot 1\cdot y_n\right] = 0$$

ie., $$(1+x^2)y_{n+2}+2nxy_{n+1}+n^2y_n-ny_n+xy_{n+1}+ny_n = 0$$

Thus $$\mathbf{(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n = 0}$$

Note : ***Alternative version of the problem.*** *The result of this problem also holds good if* $y = \sinh^{-1}x$ since $\log(x+\sqrt{1+x^2}) = \sinh^{-1}x$

---

19. *If* $y = e^{m\cos^{-1}x}$ *prove that* $(1-x^2)y_2-xy_1 = m^2y$ *and hence show that* $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n = 0$

>> By data, $y = e^{m\cos^{-1}x}$

$$\therefore \quad y_1 = e^{m\cos^{-1}x}\cdot-\frac{m}{\sqrt{1-x^2}} = -\frac{my}{\sqrt{1-x^2}} \qquad \text{or} \qquad \sqrt{1-x^2}\,y_1 = -my$$

Differentiating again we get,

$$\sqrt{1-x^2}\cdot y_2+\frac{1}{2\sqrt{1-x^2}}(-2x)y_1 = -my_1$$

or $$(1-x^2)y_2-xy_1 = -m\left\{\sqrt{1-x^2}\,y_1\right\} = -m(-my) = m^2y$$

$\therefore \quad (1-x^2)\, y_2 - x\, y_1 - m^2 y = 0$

Now $\quad D^n\left[(1-x^2)\, y_2 \right] - D^n\left[x\, y_1 \right] - m^2 D^n [y] = 0$

Applying Leibnitz theorem to the product terms we have,

$$\left\{ (1-x^2)\, y_{n+2} + n \cdot (-2x) \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} (-2)\, y_n \right\}$$

$$- \left\{ x \cdot y_{n+1} + n \cdot 1 \cdot y_n \right\} - m^2 y_n = 0$$

ie., $(1-x^2)\, y_{n+2} - 2nx\, y_{n+1} - n^2 y_n + n y_n - x\, y_{n+1} - n y_n - m^2 y_n = 0$

Thus $\quad (1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - (m^2+n^2)\, y_n = 0$

---

20. *If* $x = \sin t$ *and* $y = \cos mt$, *prove that*

$$(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} + (m^2 - n^2)\, y_n = 0.$$

\>\> By data $x = \sin t$ and $y = \cos mt$

$x = \sin t \Rightarrow t = \sin^{-1} x$ and $y = \cos mt$ becomes

$y = \cos (m \sin^{-1} x)$.

Differentiating *w.r.t* x we get

$$y_1 = -\sin (m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}} \quad \text{or} \quad \sqrt{1-x^2}\, y_1 = -m \sin (m \sin^{-1} x)$$

Differentiating again *w.r.t* x we get,

$$\sqrt{1-x^2}\, y_2 + \frac{1}{2\sqrt{1-x^2}} (-2x)\, y_1 = -m \cos (m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

or $\quad (1-x^2)\, y_2 - x\, y_1 = -m^2 y \quad$ or $\quad (1-x^2)\, y_2 - x\, y_1 + m^2 y = 0$

Note : *This result is almost same as that of the previous example and proceeding on the same lines we can arrive at the desired result.*

Thus $\quad (1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} + (m^2 - n^2)\, y_n = 0.$

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21. *If* $y^{1/m} + y^{-1/m} = 2x$, *show that*

$$(x^2 - 1)\, y_{n+2} + (2n+1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0$$

\>\> By data, $y^{1/m} + y^{-1/m} = 2x$

ie., $\quad y^{1/m} + \dfrac{1}{y^{1/m}} = 2x \quad$ or $\quad (y^{1/m})^2 + 1 = 2x (y^{1/m})$

ie., $(y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0$ which is a quadratic equation in $y^{1/m}$.

If $t = y^{1/m}$ the equation assumes the form :

$$t^2 - 2xt + 1 = 0$$

We shall solve for t using the quadratic formula.

$$\therefore \quad t = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

ie., $$t = \frac{2x \pm 2\sqrt{x^2-1}}{2} \quad \text{or} \quad y^{1/m} = x \pm \sqrt{x^2-1}$$

Let $y^{1/m} = x + \sqrt{x^2-1}$

$$\Rightarrow \quad y = \left[x + \sqrt{x^2-1}\right]^m$$

We now proceed to obtain a relation in y, y_1, y_2

Taking logarithms on both sides we get,

$$\log y = m \log\left[x + \sqrt{x^2-1}\right].$$

Differentiating *w.r.t* x we get,

$$\frac{1}{y} y_1 = m \cdot \frac{1}{[x + \sqrt{x^2-1}]}\left\{1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x\right\}$$

ie., $$\frac{1}{y} y_1 = m \cdot \frac{1}{[x + \sqrt{x^2-1}]} \frac{[\sqrt{x^2-1} + x]}{\sqrt{x^2-1}} = \frac{m}{\sqrt{x^2-1}}$$

Also if, $y = [x - \sqrt{x^2-1}]^m$ we obtain

$$\frac{1}{y} y_1 = \frac{-m}{\sqrt{x^2-1}}$$

Thus $$\frac{1}{y} y_1 = \pm \frac{m}{\sqrt{x^2-1}}$$

Squaring and cross multiplying we get

$$(x^2-1) y_1^2 = m^2 y^2.$$

Differentiating *w.r.t* x again we get,

$$(x^2-1) 2y_1 y_2 + 2x y_1^2 = m^2 (2 y y_1)$$

or $$(x^2-1) y_2 + x y_1 - m^2 y = 0, \text{ on dividing by } 2y_1.$$

Now differentiating each term n times we have,

$$D^n\left[(x^2-1)y_2\right]+D^n[xy_1]-m^2D^n[y]=0$$

Applying Leibnitz theorem to the product terms we get,

$$\left\{(x^2-1)y_{n+2}+n\cdot 2x\cdot y_{n+1}+\frac{n(n-1)}{1\cdot 2}\cdot 2y_n\right\}$$
$$+\left\{x\cdot y_{n+1}+n\cdot 1\cdot y_n\right\}-m^2y_n=0$$

ie., $(x^2-1)y_{n+2}+2nxy_{n+1}+n^2y_n-ny_n+xy_{n+1}+ny_n-m^2y_n=0$

Thus $\quad (x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$

---

22. *If* $y=a(x+\sqrt{x^2-1})^n+b(x-\sqrt{x^2-1})^n$, *prove that*
$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}=0.$

>> By data, $y=a(x+\sqrt{x^2-1})^n+b(x-\sqrt{x^2-1})^n$

$$\therefore \quad y_1=a\cdot n(x+\sqrt{x^2-1})^{n-1}\left\{1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right\}$$
$$+b\cdot n(x-\sqrt{x^2-1})^{n-1}\left\{1-\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right\}$$

ie., $$y_1=an(x+\sqrt{x^2-1})^{n-1}\frac{(\sqrt{x^2-1}+x)}{\sqrt{x^2-1}}+bn(x-\sqrt{x^2-1})^{n-1}\frac{(\sqrt{x^2-1}-x)}{\sqrt{x^2-1}}$$

or $$\sqrt{x^2-1}\,y_1=an(x+\sqrt{x^2-1})^n-bn(x-\sqrt{x^2-1})^n$$

Differentiating again *w.r.t* x and simplifying as before we get,

$$\sqrt{x^2-1}\cdot y_2+\frac{1}{2\sqrt{x^2-1}}\cdot 2xy_1$$
$$=\frac{an^2(x+\sqrt{x^2-1})^n+bn^2(x-\sqrt{x^2-1})^n}{\sqrt{x^2-1}}$$

or $\quad (x^2-1)y_2+xy_1=n^2y$

Now, $\quad D^n[(x^2-1)y_2]+D^n[xy_1]=n^2y_n$

Applying Leibnitz theorem to each term in the L.H.S we get,

$$\left\{(x^2-1)y_{n+2}+n\cdot 2x\cdot y_{n+1}+\frac{n(n-1)}{1\cdot 2}\cdot 2y_n\right\}$$
$$+\left\{xy_{n+1}+n\cdot 1\cdot y_n\right\}-n^2y_n=0$$

ie., $(x^2-1)\,y_{n+2}+2\,n\,x\,y_{n+1}+n^2 y_n-ny_n+x\,y_{n+1}+ny_n-n^2\,y_n=0$

Thus $\mathbf{(x^2-1)\,y_{n+2}+(2\,n+1)\,x\,y_{n+1}=0}$

---

23. *If* $y=\sin^{-1}x$, *prove that*

$(1-x^2)\,y_{n+2}-(2\,n+1)\,x\,y_{n+1}-n^2\,y_n=0$. *Hence show that*

$$y_n(0)=\begin{cases}0 & \text{when } n \text{ is even}\\ (n-2)^2\,(n-4)^2\cdots 5^2\cdot 3^2\cdot 1^2 & \text{when } n \text{ is odd}\end{cases}$$

>> By data $y=\sin^{-1}x$

$\therefore\quad y_1=\dfrac{1}{\sqrt{1-x^2}}$ or $\sqrt{1-x^2}\,y_1=1.$

Differentiating *w.r.t x* we get

$$\sqrt{1-x^2}\cdot y_2+\frac{1}{2\sqrt{1-x^2}}(-2x)\,y_1=0 \quad \text{or} \quad (1-x^2)\,y_2-x\,y_1=0.$$

Differentiating n times by applying Leibnitz theorem we get,

$$\left\{(1-x^2)\,y_{n+2}+n\cdot(-2x)\cdot y_{n+1}+\frac{n(n-1)}{1\cdot 2}(-2)\,y_n\right\}-\left\{x\,y_{n+1}+n\cdot 1\cdot y_n\right\}=0$$

or $(1-x^2)\,y_{n+2}-2\,n\,x\,y_{n+1}-n^2\,y_n+ny_n-x\,y_{n+1}-ny_n=0$

Thus $\mathbf{(1-x^2)\,y_{n+2}-(2\,n+1)\,x\,y_{n+1}-n^2\,y_n=0}$

Now putting $x=0$ we get $y_{n+2}(0)=n^2\,y_n(0)$ ... (1)

If $n=0: y_2(0)=0^2\,y_0(0)=0$

If $n=2: y_4(0)=2^2\,y_2(0)=0$ etc.

Thus $y_2(0)=0,\ y_4(0)=0,\ y_6(0)=0$

$\Rightarrow$ $\mathbf{y_n(0)=0}$ **when n is even.**

Also $y_1=\dfrac{1}{\sqrt{1-x^2}}$ and $y_1(0)=1$

If $n=1$ in (1) : $y_3(0)=1^2\,y_1(0)$

If $n=3$ in (1) : $y_5(0)=3^2\,y_3(0)$

If $\quad n = 5$ in (1) : $y_7(0) = 5^2 y_5(0)$

. . . .

. . . .

$$y_{n-2}(0) = (n-4)^2 y_{n-4}(0)$$

$$y_n(0) = (n-2)^2 y_{n-2}(0)$$

Thus by back substitiution we get when n is odd,

$$\boldsymbol{y_n(0) = (n-2)^2(n-4)^2 \cdots 5^2 \cdot 3^2 \cdot 1^2} \text{ since } y_1(0) = 1$$

---

24. *If* $x = \tan(\log y)$, *find the value of*

$$(1+x^2)\,y_{n+1} + (2nx-1)\,y_n + n(n-1)\,y_{n-1}$$

\>> By data $x = \tan(\log y) \Rightarrow \tan^{-1} x = \log y$ or $y = e^{\tan^{-1} x}$.

Since the desired relation involves y_{n+1}, y_n and y_{n-1} we can find y_1 and differentiate n times the result associated with y_1 and y.

Consider $y = e^{\tan^{-1} x}$ $\therefore$ $y_1 = e^{\tan^{-1} x} \cdot \dfrac{1}{1+x^2}$ or $(1+x^2)\,y_1 = y$

Differentiating n times we have

$$D^n[(1+x^2)\,y_1] = D^n[y]$$

Applying Leibnitz theorem onto L.H.S, we have,

$$\left\{(1+x^2)\,D^n(y_1) + n \cdot 2x \cdot D^{n-1}(y_1) + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot D^{n-2}(y_1)\right\} = y_n$$

ie., $\quad (1+x^2)\,y_{n+1} + 2nx\,y_n + n(n-1)\,y_{n-1} - y_n = 0$

Thus $\quad \boldsymbol{(1+x^2)\,y_{n+1} + (2nx-1)\,y_n + n(n-1)\,y_{n-1} = 0}$

---

25. *If* $y = e^{x^2/2}\cos x$, *show that*

$$y_{2n+2}(0) - 4n\,y_{2n}(0) + 2n(2n-1)\,y_{2n-2}(0) = 0$$

\>> By data $y = e^{x^2/2}\cos x$

$\therefore \quad y_1 = e^{x^2/2}(-\sin x) + x\,e^{x^2/2}\cos x$

ie., $\quad y_1 = -e^{x^2/2}\sin x + x\,y \qquad \ldots (1)$

Differentiating again,

$$y_2 = -\left[e^{x^2/2} \cos x + x\, e^{x^2/2} \sin x \right] + \left[x\, y_1 + y \right]$$

ie., $$y_2 = -y - x\, e^{x^2/2} \sin x + x\, y_1 + y$$

ie., $$y_2 = x\,(y_1 - x\,y) + x\,y_1 \text{ by using (1).}$$

or $$y_2 - 2\,x\,y_1 + x^2\,y = 0 \qquad \ldots (2)$$

We have to now differentiate this relation 2 n times and in respect of second and third terms we have to employ Leibnitz theorem in the form,

$$(uv)_{2n} = uv_{2n} + 2\,n_{C_1}\,u_1\,v_{2n-1} + 2\,n_{C_2}\,u_2\,v_{2n-2} + \cdots + u_{2n}\,v$$

From (2) we have $D^{2n}(y_2) - 2\,D^{2n}(x\,y_1) + D^{2n}(x^2\,y) = 0$

$$\therefore \quad y_{2n+2} - 2\left\{ x\,D^{2n}(y_1) + 2\,n \cdot 1 \cdot D^{2n-1}(y_1) \right\}$$
$$+ \left\{ x^2\,D^{2n}(y) + 2\,n \cdot 2\,x \cdot D^{2n-1}(y) + \frac{2\,n\,(2\,n-1)}{1 \cdot 2} \cdot 2 \cdot D^{2n-2}(y) \right\} = 0$$

ie., $$y_{2n+2} - 2\,x\,y_{2n+1} - 4\,n\,y_{2n} + x^2\,y_{2n} + 4n\,x\,y_{2n-1} + 2\,n\,(2\,n-1)\,y_{2n-2} = 0$$

Now putting $x = 0$ throughout we get,

$$\mathbf{y_{2n+2}(0) - 4n\,y_{2n}(0) + 2\,n\,(2\,n-1)\,y_{2n-2}(0) = 0}$$

---

26. *If* $y = a\,x^{n+1} + \dfrac{b}{x^n}$ *prove that* $x^2\,y_2 = n\,(n+1)\,y$ *& differentiate this result* n *times.*

\>> By data, $y = a\,x^{n+1} + b\,x^{-n}$

$$\therefore \quad y_1 = a\,(n+1)\,x^n + b\,(-n)\,x^{-n-1}$$

$$y_2 = a\,(n+1)\,n\,x^{n-1} + b\,(-n)\,(-n-1)\,x^{-n-2}$$

or $$y_2 = a\,(n+1)\,n\,x^{n-1} + b\,n\,(n+1)\,x^{-n-2}$$

Now $$x^2\,y_2 = n\,(n+1)\left[a\,x^{n+1} + b\,x^{-n} \right]$$

ie., $$x^2\,y_2 = n\,(n+1)\,y$$

Differentiating this result n times we have

$$D^n(x^2\,y_2) - n\,(n+1)\,D^n(y) = 0$$

ie., $\left\{ x^2 y_{n+2} + n \cdot 2x \cdot y_{n+1} + \dfrac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n \right\} - n(n+1) y_n = 0$

ie., $x^2 y_{n+2} + 2nx y_{n+1} + n^2 y_n - n y_n - n^2 y_n - n y_n = 0$

Thus $\mathbf{x^2 y_{n+2} + 2nx y_{n+1} - 2n y_n = 0}$

---

27. *If* $y = (ax+b)^m$ *show that if* $m > n$, $y_n = \dfrac{m!}{(m-n)!} a^n (ax+b)^{m-n}$. *Interpret the result for the case* $m < n$ *and* $m = n$. *Hence show that*

$D^{2n}[(x^2+1)^n] = 2n!$

\>> By data $y = (ax+b)^m$ where $m > n$

Differentiating successively we obtain,

$$y_n = m(m-1)(m-2) \cdots [m-(n-1)] a^n (ax+b)^{m-n} \quad \ldots (\textit{Result } F_3)$$

Multiplying and dividing by the term $(m-n)!$ which is given by

$(m-n)(m-n-1)(m-n-2) \cdots 3 \cdot 2 \cdot 1$

$[\because m > n \Rightarrow (m-n) > 0$ *and if* $(m-n) = k$, *a positive integer then* $k! = k(k-1)(k-2) \cdots 3 \cdot 2 \cdot 1]$

Thus $y_n = \{m(m-1)(m-2) \cdots m-(n-1)\}$

$$\times \frac{\{(m-n)(m-n-1) \cdots 3 \cdot 2 \cdot 1\}}{(m-n)!} \cdot a^n \cdot (ax+b)^{m-n}$$

Combining all the terms in the numerator, the same represents the product of first m natural numbers which is $m!$

Thus if $m > n$ $\quad \mathbf{y_n = \dfrac{m!}{(m-n)!} a^n (ax+b)^{m-n}} \quad \ldots (1)$

Next if $m = n$ $\quad D^n\left[(ax+b)^m\right] = \dfrac{n!}{0!} a^n (ax+b)^0$

$\therefore \quad y_n = n!\, a^n = \text{constant}$, when $m = n$

Thus if $m = n$, $\quad \mathbf{D^n\left[(ax+b)^n\right] = n!\, a^n}$

Also if $m < n$ then $D^n\left[(ax+b)^m\right] = D^n(\text{constant}) = 0$

$\because \quad D^{n-m}\left[D^m (ax+b)^m\right] = D^k(\text{constant}) = 0$ since $k = n - m > 0$

Thus if $m < n$ $\quad \mathbf{D^n\left[(ax+b)^m\right] = 0}$

Also we have to find $D^{2n}\left[(x^2+1)^n \right]$ using the above results.

Expanding $(x^2+1)^n$ by the binomial theorem we obtain,

$$(x^2+1)^n = (x^2)^n + {}^nC_1 (x^2)^{n-1} \cdot 1 + {}^nC_2 (x^2)^{n-2} \cdot 1^2 + \cdots + 1$$

ie., $$(x^2+1)^n = x^{2n} + {}^nC_1 x^{2n-2} + {}^nC_2 x^{2n-4} + \cdots + 1$$

$$\therefore \quad D^{2n}\left[(x^2+1)^n \right] = D^{2n}(x^{2n}) + {}^nC_1 D^{2n}(x^{2n-2})$$

$$+ {}^nC_2 D^{2n}(x^{2n-4}) + \cdots + 0 \qquad \ldots (4)$$

We have (2) and (3) when $a = 1, b = 0$ in the form

$$D^n(x^n) = n! \text{ and } D^n(x^m) = 0 \text{ if } m < n$$

Using these results in (4) we get,

$$D^{2n}\left[(x^2+1)^n \right] = 2n! + {}^nC_1 \cdot 0 + {}^nC_2 \cdot 0 + \cdots = 2n!$$

Thus $$D^{2n}\left[(x^2+1)^n \right] = 2n!$$

---

28. *If* $x + y = 1$ *then show that.*

$$\frac{d^n}{dx^n}[x^n y^n] = n!\left\{ y^n - ({}^nC_1)^2 y^{n-1} x + ({}^nC_2)^2 y^{n-2} x^2 + \cdots + (-1)^n x^n \right\}$$

\>> By data $x + y = 1, \quad \therefore \quad y = 1 - x$

Let us take $u = x^n$ and $v = y^n = (1-x)^n$

We shall find $u_1, u_2, \cdots u_{n-2}, u_{n-1} u_n$ and $v_1, v_2, \cdots v_{n-2}, v_{n-1}, v_n$ with the aim to apply Leibnitz theorem for the product $u\,v$.

Consider $u = x^n$ and its successive derivatives are as follows.

$$u_1 = n\,x^{n-1} \; ; \; u_2 = n(n-1)x^{n-2} \text{ etc.}$$

$$u_{n-2} = n(n-1)(n-2)\cdots 3x^2 = \frac{n!}{2}x^2$$

$$u_{n-1} = n(n-1)(n-2)\cdots 3 \cdot 2x = n!\,x$$

$$u_n = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$$

Now consider $v = y^n$ where $y = 1 - x$.

$$v_1 = n\,y^{n-1}\cdot y' = n\,y^{n-1}(-1),\quad v_2 = n(n-1)\,y^{n-2}(-1)^2 \text{ etc.}$$

$$v_{n-2} = \frac{n!}{2}y^2(-1)^{n-2},\quad v_{n-1} = n!\,y(-1)^{n-1},\quad v_n = n!(-1)^n$$

We have Leibnitz theorem,

$$D^n(uv) = uv_n + {}^nC_1\,u_1 v_{n-1} + {}^nC_2\,u_2 v_{n-2}$$

$$+\cdots + {}^nC_{n-2}\,u_{n-2}v_2 + {}^nC_{n-1}\,u_{n-1}v_1 + u_n v$$

$$\therefore\quad D^n(x^n y^n) = x^n\cdot n!(-1)^n + {}^nC_1\cdot n\,x^{n-1}\cdot n!\,y(-1)^{n-1}$$

$$+ {}^nC_2\cdot n(n-1)\,x^{n-2}\cdot\frac{n!}{2}y^2(-1)^{n-2}$$

$$+\cdots + {}^nC_{n-2}\cdot\frac{n!}{2}x^2\cdot n(n-1)\,y^{n-2}(-1)^2 + {}^nC_{n-1}\cdot n!\,x\cdot(-ny^{n-1}) + n!\,y^n$$

We know that $n = {}^nC_1 = {}^nC_{n-1}$; $\dfrac{n(n-1)}{2} = {}^nC_2 = {}^nC_{n-2}$ etc.

$$\therefore\quad D^n(x^n y^n) = n!\Big\{(-1)^n x^n + ({}^nC_1)^2(-1)^{n-1}x^{n-1}y$$

$$+({}^nC_2)^2(-1)^{n-2}x^{n-2}y^2 + \cdots + ({}^nC_{n-2})^2(-1)^2x^2y^{n-2}$$

$$+({}^nC_{n-1})^2(-1)^1\,x\,y^{n-1} + y^n\Big\}$$

Reversing the order of the terms in R.H.S we have

$$D^n(x^n y^n) = n!\Big\{y^n - ({}^nC_1)^2\,x\,y^{n-1} + ({}^nC_2)^2x^2y^{n-2}\ldots$$

$$+({}^nC_{n-1})^2(-1)^{n-1}x^{n-1}y + (-1)^n x^n\Big\}$$

Remark : *The problem can also be worked by the method of mathematical induction.*

---

29. *If* $y = e^{-x^2}$, *show that* $y_{n+1} + 2x\,y_n + 2n\,y_{n-1} = 0$

>> $y = e^{-x^2}$

$\therefore\quad y_1 = e^{-x^2}(-2x)$ or $y_1 = -2xy$

We have, $y_1 + 2xy = 0$

Differentiating this result n times we have,

$$D^n [y_1] + 2 D^n [xy] = 0$$

ie., $y_{n+1} + 2 \{ x y_n + n \cdot 1 \cdot y_{n-1} \} = 0$

Thus $\mathbf{y_{n+1} + 2 x y_n + 2 n y_{n-1} = 0}$

---

30. *If* $y = \sin h (m \sin h^{-1} x)$, *establish the relation connecting* y_{n+2}, y_{n+1} *and* y_n

>> $y = \sin h (m \sinh^{-1} x)$

$\therefore$ $y_1 = \cos h (m \sin h^{-1} x) \cdot \dfrac{m}{\sqrt{1 + x^2}}$

or $\sqrt{1 + x^2}\, y_1 = m \cos h (m \sin h^{-1} x)$

Differentiating w.r.t x again we get,

$$\sqrt{1 + x^2}\, y_2 + \frac{1}{2 \sqrt{1 + x^2}} \cdot 2x y_1 = m \sin h (m \sin h^{-1} x) \frac{m}{\sqrt{1 + x^2}}$$

or $(1 + x^2) y_2 + x y_1 = m^2 y$

Differentiating each term n times we have,

$$D^n [(1 + x^2) y_2] + D^n [x y_1] - m^2 D^n (y) = 0$$

Applying Leibnitz theorem to the product terms we get,

$$\left\{ (1 + x^2) y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n (n - 1)}{1 \cdot 2} \cdot 2 \cdot y_n \right\} + \{ x y_{n+1} + n \cdot 1\, y_n \} - m^2 y_n = 0$$

Thus $(1 - x^2) y_{n+2} + (2n + 1) x y_{n+1} + (n^2 - m^2) y_n = 0$ **is the required relation.**

EXERCISES

Find y_n for the following functions. **[1 and 2]**

1. $y = x \cos 3x \cos 5x$ 2. $y = x^2 e^x \cos x$

3. If $y = x^2 e^x$, prove that $y_n = (x^2 + 2 n x + n^2 - n) e^x$

4. If $y = x \log \left(\dfrac{x - 1}{x + 1} \right)$, show that $D^n y = (- 1)^n (n - 2) ! \left[\dfrac{x - n}{(x - 1)^n} - \dfrac{x + n}{(x + 1)^n} \right]$

5. If $y = (\sin^{-1} x)^2$ prove that $(1-x^2)\,y_2 - x\,y_1 = 2$
and then apply Leibnitz theorem to differentiate the result n times.

6. If $y = \sin(m \sin^{-1} x)$ prove that

(a) $(1-x^2)\,y_2 - x\,y_1 + m^2\,y = 0$

(b) $(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} + (n^2 - m^2)\,y_n = 0$

7. If $y = \cos\log(x^2 - 2x + 1)$, prove that

$$(x-1)^2\,y_{n+2} + (2n+1)(x-1)\,y_{n+1} + (n^2+4)\,y_n = 0$$

8. If $y = \left[x + \sqrt{1+x^2}\right]^m$ show that,

$$(1+x^2)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2 - m^2)\,y_n = 0$$

9. If $y = \left[\log\left(x + \sqrt{x^2+a^2}\right)\right]^2$, prove that

$(1+x^2)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + n^2\,y_n = 0$. Hence deduce that :

$$y_n(0) = \begin{cases} 0, & \text{when } n \text{ is even} \\ (-1)^n \cdot 1^2 \cdot 2^2 \cdot 3^2 \cdots n^2, & \text{when } n \text{ is odd} \end{cases}$$

10. If $y = a\cos h(\log x^m) + b\sin h(\log x^m)$ show that

$$x^2\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2 - m^2)\,y_n = 0$$

ANSWERS

1. $$\frac{1}{2}\left[x \cdot 8^n \cos\left(\frac{n\pi}{2} + 8x\right) + n \cdot 8^{n-1}\cos\left((n-1)\frac{\pi}{2} + 8x\right)\right] + \frac{1}{2}\left[x \cdot 2^n \cos\left(\frac{n\pi}{2} + 2x\right) + n \cdot 2^{n-1}\cos\left((n-1)\frac{\pi}{2} + 2x\right)\right]$$

2. $$2^{n/2} e^x\left[x^2 \cos\left(n\frac{\pi}{4} + x\right) + \sqrt{2} \cdot n \cdot x\cos\left((n-1)\frac{\pi}{4} + x\right) + \frac{n(n-1)}{2}\cos\left((n-2)\frac{\pi}{4} + x\right)\right]$$

5. $$(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - n^2\,y_n = 0$$

1.2 Rolle's theorem and mean value theorems

1.21 Continuity and Differentiability

A function $f(x)$ is said to be *continuous* at a point $x = a$ if $\lim_{x \to a} f(x) = f(a)$

where $f(a)$ means, the value of $f(x)$ at $x = a$ and it should not be infinite, indeterminate, imaginary.

A function $f(x)$ is said to be *differentiable* at a point $x = a$ if $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists and is unique.

A function $f(x)$ is said to be continuous or differentiable in an interval if it is continuous or differentiable at each point of the interval. In simple words, we can say that $f(x)$ is continuous in an interval if the graph of $f(x)$ do not have any breaks in that interval. Differentiability accounts for the smoothness of the curve.

Notes and Notations

1. *If a function is differentiable then it is necessarily continuous but not conversely. That is, differentiability implies continuity.*

2. *If* $f(x)$ *and* $g(x)$ *are two continuous functions then* $kf(x)$, $kg(x)$ *[where k is a constant],* $f(x) \pm g(x)$, $f(x) \cdot g(x)$, $f(x)/g(x)$ *where* $g(x) \neq 0$ *for all x are all continuous functions.*

3. *Common functions like constant, any polynomial,* $\sin x$, $\cos x$, e^x, $\sinh x$, $\cosh x$ *etc, are continuous at all points. Also we can say that,* $\log x$ *is not continuous at* $x = 0$, $1/(x-1)$ *is not continuous at* $x = 1$, $\tan x$ *is not continuous at* $x = \pi/2$ *etc.*

Notations

1. $[a, b]$: *Closed interval a, b : It includes all the points between a & b including a & b.*

2. (a, b) : *Open interval a, b : It includes all the points between a & b excluding a & b.*

ie., $x \in [a, b] \Rightarrow a \leq x \leq b$; $x \in (a, b) \Rightarrow a < x < b$

1.22 Rolle's theorem and its geometrical interpretation

Statement : *If a function* $f(x)$ *is (i) continuous in* $[a, b]$ *(ii) differentiable in* (a, b) *(iii)* $f(a) = f(b)$ *then there exists atleast one point* c *in* (a, b) *that is* $a < c < b$ ***such that*** $f'(c) = 0$

Geometrical interpretation (meaning)

Geometrically the three conditions can be interpreted as follows :

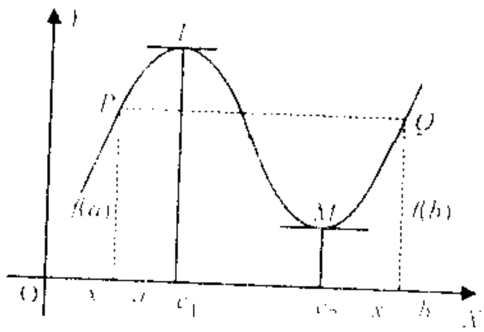
(i) The curve $f(x)$ in the interval a, b should not have any breaks including the end points.

(ii) The curve $f(x)$ is smooth and it must possess tangent at all points in the interval a, b except at the end points where tangents cannot be drawn.

(iii) The ordinates $f(a)$, $f(b)$ respectively corresponding to a, b must be of the same height from the x-axis.

When all these conditions are satisfied, geometrically the theorem means that there exists atleast one point strictly between the end points at which the tangent is parallel to the x-axis.

A picturesque illustration is as follows.



In the above figure the points P and Q correspond to $x = a$, $x = b$ and the corresponding heights $f(a)$, $f(b)$ are equal. It can be seen from the figure that there are two points L and M on the curve $[a < c_1 < b \; ; \; a < c_2 < b]$ at which the tangents are parallel to the x-axis.

Remark : *This theorem is the foundation for all other theorems to follow.*

We now proceed to establish mean value theorems by applying Rolle's theorem.

1.23 Lagrange's (first) mean value theorem

Statement : *If a function* $f(x)$ *is continuous in* $[a, b]$, *differentiable in* (a, b) *then there exists atleast one point* c *in* (a, b) *such that* $\dfrac{f(b) - f(a)}{b - a} = f'(c)$

Proof : Let us define a new function

$$\phi(x) = f(x) - k \cdot x \qquad \ldots (1)$$

where k is a constant to be chosen suitably later. Since $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) and that kx is also continuous in $[a, b]$, differentiable in (a, b) we can conclude that $\phi(x)$ is also continuous in $[a, b]$, differentiable in (a, b)

From (1) we have, $\phi(a) = f(a) - k \cdot a$; $\phi(b) = f(b) - k \cdot b$

$\therefore \quad \phi(a) = \phi(b)$ holds good if

$f(a) - k \cdot a = f(b) - k \cdot b$ or $k(b-a) = f(b) - f(a)$

$$\therefore \quad k = \frac{f(b)-f(a)}{b-a} \qquad \ldots (2)$$

Hence if k is chosen as given by (2), then $\phi(x)$ satisfy all the conditions of Rolle's theorem. Therefore by Rolle's theorem there exists atleast one point c in (a, b) such that $\phi'(c) = 0$

Differentiating (1) w.r.t. x we have, $\phi'(x) = f'(x) - k$

and $\phi'(c) = 0$ yields $f'(c) - k = 0$

ie., $k = f'(c) \qquad \ldots (3)$

Equating the R.H.S of (2) and (3) we have

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c)} \qquad \ldots (4)$$

This proves Lagrange's mean value theorem.

Note

1. *The theorem can also be put in the following forms :*

$$f(b) - f(a) = (b-a)f'(c) \quad \text{or} \quad f(b) = f(a) + (b-a)f'(c) \qquad \ldots (5)$$

2. *Further if the length of the interval $[a, b]$ is h we have $b - a = h$ or $b = a + h$. Also if we set $\theta = \frac{c-a}{h} = \frac{c-a}{b-a}$ we observe that θ lies between 0 and 1. That is $0 < \theta < 1$. Now $c = a + \theta h$ and $b = a + h$. The theorem in the form (5) becomes*

$$f(a+h) = f(a) + hf'(a+\theta h) \qquad \ldots (6)$$

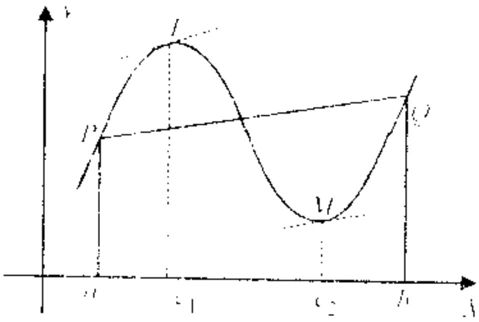
Geometrical interpretation (meaning) of the theorem

Let $P = \{a, f(a)\}$, $Q = \{b, f(b)\}$ be any two points on the curve representing $y = f(x)$.

$\therefore$ slope of $PQ = \dfrac{f(b)-f(a)}{b-a}$

As per the conditions of the theorem, the curve $f(x)$ has no breaks in the interval including the end points and possesses tangents at all points within the interval.

Geometrically the theorem means that there exists atleast one point on the curve at which the tangent is parallel to the line joining the end points. In the above figure there are two points L and M at which the tangents are parallel to the line PQ.



1.24 Cauchy's mean value theorem

Statement : *If $f(x)$ and $g(x)$ are two functions continuous in $[a, b]$, differentiable in (a, b) with $g'(x) \neq 0$ for all x in (a, b) then there exists atleast one point c in (a, b) such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof : Let us define a new function

$$\phi(x) = f(x) - k\,g(x) \qquad \ldots (1)$$

where k is a constant to be chosen suitably later. From the given conditions it is evident that $\phi(x)$ is also continuous in $[a, b]$, differentiable in (a, b)

Further from (1) we have,

$$\phi(a) = f(a) - k\,g(a)\ ;\ \phi(b) = f(b) - k\,g(b)$$

$\therefore$ $\phi(a) = \phi(b)$ holds good if

$$f(a) - k\,g(a) = f(b) - k\,g(b)$$

ie., $k[g(b) - g(a)] = f(b) - f(a)$

$$\therefore \quad k = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Here $g(b) \neq g(a)$. Because if $g(b) = g(a)$ then $g(x)$ would satisfy all the conditions of Rolle's theorem and accordingly there must exist atleast one point c in

(a, b) such that $g'(c) = 0$. This contradicts the data that $g'(x) \neq 0$ for all x in (a, b)

Hence if k is chosen as given by (2) then $\phi(x)$ satisfy all the conditions of Rolle's theorem. Therefore by Rolle's theorem there exists atleast one value c in (a, b) such that $\phi'(c) = 0$

Differentiating (1) w.r.t. x we have,

$$\phi'(x) = f'(x) - kg'(x) \quad \text{and} \quad \phi'(c) = 0 \quad \text{yields}$$

$$f'(c) - kg'(c) = 0 \qquad ie. \; f'(c) = k\,g'(c)$$

$$\therefore \qquad k = \frac{f'(c)}{g'(c)}$$

Equating the R.H.S of (2) and (3) we have

$$\boxed{\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}}$$

This proves Cauchy's mean value theorem.

Note :

1. *We can deduce Lagrange's mean value theorem from Cauchy's mean value theorem.*

Taking $g(x) = x$ *we have* $g(a) = a,\ g(b) = b$

Also $g'(x) = 1 \Rightarrow g'(c) = 1$

Hence Cauchy's mean value theorem becomes,

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(c)}{1} \quad ie., \quad \frac{f(b) - f(a)}{b - a} = f'(c)$$

This is Lagrange's mean value theorem.

2. *We can also deduce Cauchy's mean value theorem from Lagrange's theorem.*

Let us take the format of Lagrange's mean value theorem for $f(x)$ *and* $g(x)$ *in the form*

$$\frac{f(b) - f(a)}{b - a} = f'(c_1) \qquad \ldots (1)$$

and

$$\frac{g(b) - g(a)}{b - a} = g'(c_2) \qquad \ldots (2)$$

Dividing (1) by (2) we have,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c_1)}{g'(c_2)}$$

This is Cauchy's mean value theorem provided $c_1 = c_2 = c$ *where* $a < c < b$.

Note : 1. *Taking* $b-a=h$ *and* $c=a+\theta h$ *where* $\theta=\dfrac{c-a}{b-a}$ *we observe that* $0<\theta<1$. *Taylor's theorem becomes*

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots$$

$$+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h)$$

2. *By taking* $n=1$ *in both the forms of Taylor's theorem we get*

$$f(b)=f(a)+(b-a)f'(c) \quad or \quad f(a+h)=f(a)+hf'(a+\theta h)$$

These are the two forms of Lagrange's mean value theorem. Hence the same is referred to as Lagrange's first mean value theorem.

When $n=2$ *we get*

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(c).$$

This is referred to as Lagrange's second mean value theorem and so on.

... of Rolle's ... theorems

... procedure ...

- We ensure the continuity of the given function / functions in the interval. $f(a)=f(b)$ also has to be ensured for Rolle's theorem.
- The derivative of the given function / functions must be found and we ensure differentiatiability in the interval.
- We need to identify the value / values of c belonging to the open interval satisfying the relevant theorem.

>> $f(x) = x^2$ is continuous in $[-1, 1]$ & $f'(x) = 2x$ exist for all values in $(-1, 1)$ $f(-1) = (-1)^2 = 1$ and $f(1) = 1^2 = 1$ $\therefore$ $f(-1) = f(1)$

Hence all the three conditions of the theorem are satisfied.

Now consider $f'(c) = 0$ that is $2c = 0$ or $c = 0$
$\mathbf{c = 0 \in (-1, 1)}$ and hence Rolle's theorem is verified.

Geometrically $f(x) = y = x^2$ is a parabola symmetrical about the y-axis passing through the origin and that the x-axis itself is the tangent to the curve at $x = c = 0$

32. $f(x) = x^2(1 - 2x + x^2) = x^2 - 2x^3 + x^4$

$f(x)$ is continuous in $[0, 1]$

$f'(x) = 2x - 6x^2 + 4x^3$ exists in $(0, 1)$

Also $f(0) = 0 = f(1)$

Hence all the conditions of the theorem are satisfied.

Now consider $f'(c) = 0$

ie., $2c - 6c^2 + 4c^3 = 0$ *or* $2c(1 - 3c + 2c^2) = 0$

ie., $2c(2c - 1)(c - 1) = 0 \Rightarrow 2c = 0,\ 2c - 1 = 0,\ c - 1 = 0$

$\therefore$ $c = 0 \quad c = 1/2 \quad c = 1$

$\mathbf{c = 1/2 \in (0, 1)}$. **Thus Rolle's theorem is verified**

33. $f(x) = (x^2 + 2x)e^{-x/2}$ is continuous in $[-2, 0]$

$f'(x) = (x^2 + 2x)e^{-x/2}(-1/2) + (2x + 2)e^{-x/2}$

ie., $f'(x) = \frac{e^{-x/2}}{2}(-x^2 - 2x + 4x + 4)$

$\therefore \quad f'(x) = \frac{e^{-x/2}}{2}(-x^2 + 2x + 4)$

$f'(x)$ exists in $(-2, 0)$ Also $f(-2) = 0 = f(0)$

Hence all the conditions of the theorem are satisfied.

Now consider $f'(c) = 0$

From (1) we have $\frac{e^{-c/2}}{2}(-c^2 + 2c + 4) = 0$

Since $e^{-c/2}$ cannot be zero, we must have

$-c^2 + 2c + 4 = 0$ or $c^2 - 2c - 4 = 0$

$\therefore \quad c = \frac{2 \pm \sqrt{4+16}}{2} = \frac{2 \pm \sqrt{20}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$

$c = 1 - \sqrt{5} \approx -1.236 \in (-2, 0)$. **Thus Rolle's theorem is verified.**

34. $f(x) = (x-a)^p (x-b)^q$ is continuous in $[a, b]$

$f'(x) = (x-a)^p q (x-b)^{q-1} + p(x-a)^{p-1}(x-b)^q$

$= (x-a)^{p-1}(x-b)^{q-1}[q(x-a) + p(x-b)]$

$f'(x) = (x-a)^{p-1}(x-b)^{q-1}[(q+p)x - (qa+pb)]$ ... (1)

$f'(x)$ exists in (a, b)

Also $f(a) = 0 = f(b)$

Hence all the conditions of the theorem are satisfied.

Now Consider $f'(c) = 0$

From (1), $(c-a)^{p-1}(c-b)^{q-1}[(q+p)c - (qa+pb)] = 0$

$\Rightarrow \quad c - a = 0, \ c - b = 0, \ (q+p)c - (qa+pb) = 0$

ie., $c = a, \ c = b, \ c = \frac{pb+qa}{p+q}$

a, b are the end points. $c = pb + qa / p + q$ is the x-coordinate of the point which divides the line joining $[a, f(a)]$, $[b, f(b)]$ internally in the ratio $p : q$

$\therefore \quad c = pb + qa / p + q \in (a, b)$. **Thus Rolle's theorem is verified.**

35. $f(x) = e^x(\sin x - \cos x)$ is continuous in $[\pi/4, 5\pi/4]$

$f'(x) = e^x(\cos x + \sin x) + e^x(\sin x - \cos x)$

$f'(x) = 2e^x \sin x$ . . . (1)

$f(x)$ is differentiable in $(\pi/4, 5\pi/4)$

$f(\pi/4) = e^{\pi/4}(\sin \pi/4 - \cos \pi/4) = e^{\pi/4}(1/\sqrt{2} - 1/\sqrt{2}) = 0$

$f(5\pi/4) = e^{5\pi/4}(\sin 5\pi/4 - \cos 5\pi/4)$
$= e^{5\pi/4}(-1/\sqrt{2} + 1/\sqrt{2}) = 0$

$\therefore$ $f(\pi/4) = 0 = f(5\pi/4)$

Hence all the conditions of the theorem are satisfied.

Now consider $f'(c) = 0$

From (1) we have, $2e^c \sin c = 0$. But $e^c \neq 0$

$\therefore$ $\sin c = 0 \Rightarrow c = n\pi$ where $n = 0, 1, 2, \ldots$

But $c = \pi \in (\pi/4, 5\pi/4)$. **Thus Rolle's theorem is verified.**

36. $f(x) = \dfrac{\sin 2x}{e^{2x}}$ is continuous in $[0, \pi/2]$

$$f'(x) = \frac{e^{2x} \cdot 2\cos 2x - \sin 2x \cdot 2e^{2x}}{(e^{2x})^2} = \frac{2e^{2x}(\cos 2x - \sin 2x)}{(e^{2x})^2}$$

ie., $f'(x) = \dfrac{2(\cos 2x - \sin 2x)}{e^{2x}}$ . . . (1)

$f'(x)$ exists in $(0, \pi/2)$

Also $f(0) = 0 = f(\pi/2)$ $\because$ $\sin 0 = 0 = \sin \pi$

Hence all the conditions of the theorem are satisfied.

Now consider $f'(c) = 0$

We have from (1)

$$\frac{2(\cos 2c - \sin 2c)}{e^{2c}} = 0$$

$\Rightarrow$ $\cos 2c - \sin 2c = 0$

ie., $\cos 2c = \sin 2c$ or $\tan 2c = 1 \Rightarrow 2c = \pi/4$

$\therefore$ $c = \pi/8 \in (0, \pi/2)$. **Thus Rolle's theorem is verified.**

>> The given $f(x)$ is continuous in $[a, b]$ since $0 < a < b$

The given $f(x)$ can be written in the form

$$f(x) = \log(x^2 + ab) - \log(a+b) - \log x$$

$$\therefore \quad f'(x) = \frac{2x}{x^2 + ab} - 0 - \frac{1}{x} = \frac{2x^2 - (x^2 + ab)}{(x^2 + ab)x}$$

ie., $$f'(x) = \frac{x^2 - ab}{(x^2 + ab)x} \quad \ldots (1)$$

$f'(x)$ exists in (a, b)

Also $$f(a) = \log\left[\frac{a^2 + ab}{(a+b)a}\right] = \log 1 = 0$$

$$f(b) = \log\left[\frac{b^2 + ab}{(a+b)b}\right] = \log 1 = 0$$

$\therefore \quad f(a) = f(b)$

Hence all the conditions of the theorem are satisfied.

Now consider $f'(c) = 0$

From (1) we have, $\dfrac{c^2 - ab}{(c^2 + ab)c} = 0 \Rightarrow c^2 - ab = 0$ ie., $c = \pm\sqrt{ab}$

$\therefore \quad c = +\sqrt{ab} \in (a, b)$ and we know that $\sqrt{ab}$ is the geometric mean of a and b.

Remark : *Consider the function $f(x) = x^3$ in $[-1, 1]$. $f(x)$ is continuous in $[-1, 1]$, $f'(x) = 3x^2$ exists in $(-1, 1)$, $f(-1) \neq f(1)$. But $f'(c) = 0$ gives $3c^2 = 0$ or $c = 0 \in (-1, 1)$ Though one of the conditions of the theorem is not satisfied we have a value $c = 0 \in (-1, 1)$ such that $f'(c) = 0$. Such situations are interpreted as follows.*

When all the conditions of the theorem are satisfied ***definitely*** *we will have atleast one value 'c' within the interval such that $f'(c) = 0$.*

38. We have Lagrange's mean value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Here $f(x) = \log x$; $a = 1$, $b = e$

$f(x)$ is continuous in $[1, e]$ Also $f'(x) = 1/x$

$\therefore$ $f(x)$ is differentiable in $(1, e)$

Hence the theorem becomes

$$\frac{f(e)-f(1)}{e-1} = \frac{1}{c} \quad ie., \quad \frac{\log e - \log 1}{e-1} = \frac{1}{c}$$

But $\log 1 = 0$, $\log e = 1$ and hence we have $\dfrac{1}{e-1} = \dfrac{1}{c}$

or $\mathbf{c = e - 1 \approx 2.7 - 1 = 1.7 \in (1, e)}$. **Thus the theorem is verified.**

39. We have Lagrange's mean value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

The given $f(x)$ is continuous in $[0, 4]$

$f(x) = (x-1)(x-2)(x-3)$; $a = 0$, $b = 4$ by data.

$\therefore$ $f(b) = f(4) = 3.2.1 = 6$ and

$f(a) = f(0) = (-1)(-2)(-3) = -6$

We have $f(x) = x^3 - 6x^2 + 11x - 6$ in the simplified form.

$\therefore$ $f'(x) = 3x^2 - 12x + 11$ exists in $(0, 4)$

The theorem becomes

$$\frac{f(4)-f(0)}{4-0} = 3c^2 - 12c + 11$$

ie., $\dfrac{6-(-6)}{4} = 3c^2 - 12c + 11$

or $3c^2 - 12c + 8 = 0$

$\therefore$ $c = \dfrac{12 \pm \sqrt{144-96}}{6} = \dfrac{12 \pm \sqrt{48}}{6}$; $c = \dfrac{12+\sqrt{48}}{6}, \dfrac{12-\sqrt{48}}{6}$

$c \approx 3.15$ and 0.85 both belong to $(0, 4)$. Thus the theorem is verified.

40. We have the theorem $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

$f(x) = x^3 - 3x^2 + 2x$. $f(x)$ is continuous in $[0, 1/2]$

$f'(x) = 3x^2 - 6x + 2$ $\therefore$ $f(x)$ is differentiable in $(0, 1/2)$

With $a = 0$, $b = 1/2$ and $f(x) = x(x-1)(x-2)$ the theorem becomes

$$\frac{1/2(1/2-1)(1/2-2)-0}{1/2-0} = 3c^2 - 6c + 2 \quad \text{or}$$

$$(-1/2)(-3/2) = 3c^2 - 6c + 2$$

ie., $12c^2 - 24c + 5 = 0$ $\therefore$ $c = \dfrac{24 \pm \sqrt{576-240}}{24} = \dfrac{24 \pm \sqrt{336}}{24}$

ie., $c \approx 1.76$ and 0.24

Here **$c = 0.24 \in (0, 1/2)$. Thus the theorem is verified.**

41. We have the theorem $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

$f(x) = \cos^2 x$ is continuous in $[0, \pi/2]$

$f'(x) = -2\cos x \sin x = -\sin 2x$ exists in $(0, \pi/2)$

$f(b) = f(\pi/2) = \cos^2(\pi/2) = 0$; $f(a) = f(0) = \cos^2 0 = 1$

$\therefore$ the theorem becomes

$$\frac{0-1}{\pi/2-0} = -\sin 2c \quad \text{or} \quad \sin 2c = 2/\pi \Rightarrow 2c = \sin^{-1}(2/\pi)$$

$\therefore$ $c = (1/2)\cdot\sin^{-1}(2/\pi) \approx 0\cdot345$; But $\pi/2 \approx 1\cdot57$

Here **$c = 0\cdot345 \in (0, 1\cdot57)$. Thus the theorem is verified.**

42. We have the theorem $\frac{f(b)-f(a)}{b-a}=f'(c)$

$f(x) = \tan^{-1} x$ is continuous in $[0, 1]$

$f'(x) = 1/1+x^2$ exists in $(0, 1)$

$f(b) = f(1) = \tan^{-1}(1) = \pi/4$; $f(a) = f(0) = \tan^{-1} 0 = 0$

$\therefore$ the theorem becomes

$$\frac{\pi/4-0}{1} = \frac{1}{1+c^2} \quad \text{or} \quad \pi(1+c^2) = 4$$

ie., $\pi c^2 = 4-\pi$ or $c^2 = (4-\pi)/\pi$

$\therefore$ $c = \sqrt{(4-\pi)/\pi}$

$c \approx 0 \cdot 523 \in (0, 1)$. **Thus the theorem is verified.**

>> We have the theorem in the form

$$f(b) = f(a)+(b-a)f'(a+\theta h) \qquad \ldots (1)$$

$f(x) = e^x$ is continuous in $[0, 1]$ $f'(x) = e^x$ exists in $(0, 1)$

Here $a=0,\ b=1$ $\therefore$ $f'(a+\theta h) = f'(\theta h) = e^{\theta h}$

$f(b) = f(1) = e^1 = e$; $f(a) = f(0) = e^0 = 1$

Hence (1) becomes,

$e = 1+(1-0)e^{\theta h}$ *ie.,* $e-1 = e^{\theta h}$

But $h = b-a = 1$ $\therefore$ $e^{\theta} = e-1 \Rightarrow \theta = \log(e-1)$

ie., $\theta = \log(e-1) \approx \log_e(2.7-1) = \log_e(1.7) \approx 0.53 < 1$

Thus the required $\theta = 0.53$

44. We have Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \quad \text{Here } a = 0,\ b = 16$$

Let $f(x) = \sqrt{x+9}$ $\quad g(x) = \sqrt{x}$

$\therefore$ $f'(x) = \dfrac{1}{2\sqrt{x+9}}$ $\qquad g'(x) = \dfrac{1}{2\sqrt{x}}$

$f(x)$ and $g(x)$ are continuous in $[0, 16]$, differentiable in $(0, 16)$. $g'(x) \neq 0 \ \forall \ x \in (0, 16)$

Hence the theorem becomes

$$\frac{f(16)-f(0)}{g(16)-g(0)} = \frac{1/2\sqrt{c+9}}{1/2\sqrt{c}}$$

ie., $\dfrac{\sqrt{25}-\sqrt{9}}{\sqrt{16}-\sqrt{0}} = \dfrac{\sqrt{c}}{\sqrt{c+9}}$ *ie.,* $\dfrac{5-3}{4} = \dfrac{\sqrt{c}}{\sqrt{c+9}}$

ie., $\dfrac{1}{2} = \dfrac{\sqrt{c}}{\sqrt{c+9}}$ $\qquad \therefore \ 4c = c+9$ or $c = 3$

The value $c = 3 \in (0, 16)$. **Thus the theorem is verified.**

---

45. We have Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \quad \text{where } a = 3,\ b = 7 \text{ by data.}$$

$f(x) = e^x$ gives $f'(x) = e^x$; $g(x) = e^{-x}$ gives $g'(x) = -e^{-x}$

$f(x)$, $g(x)$ are continuous in $[3, 7]$, differentiable in $(3, 7)$
$g'(x) \neq 0 \ \forall \ x \in (3, 7)$

Hence the theorem becomes

$$\frac{f(7)-f(3)}{g(7)-g(3)} = \frac{e^c}{-e^{-c}}$$

ie., $\dfrac{e^7-e^3}{\dfrac{1}{e^7}-\dfrac{1}{e^3}} = -e^{2c}$ or $\dfrac{(e^7-e^3)e^{10}}{(e^3-e^7)} = -e^{2c}$

ie., $\dfrac{(e^3-e^7)e^{10}}{(e^3-e^7)} = e^{2c} \Rightarrow e^{2c} = e^{10} \Rightarrow 2c = 10$ or $c = 5$

$c = 5 \in (3, 7)$. **Thus the theorem is verified.**

---

46. $f(x) = \log x$. Let $g(x) = f'(x) = 1/x$

$\therefore \quad f'(x) = 1/x \;;\; g'(x) = -1/x^2$ Also $a = 1,\ b = e$

$f(x)$ and $g(x)$ are continuous in $[1, e]$, differentiable in $(1, e)$.

$$g'(x) \neq 0 \quad \forall \ x \in (1, e)$$

We have Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

ie.,
$$\frac{f(e)-f(1)}{g(e)-g(1)} = \frac{1/c}{-1/c^2}$$

ie.,
$$\frac{\log e - \log 1}{(1/e)-1} = -c \quad \text{or} \quad \frac{1-0}{(1-e)/e} = -c \quad \text{or} \quad \frac{e}{1-e} = -c$$

ie.,
$$c = \frac{e}{e-1} \approx \frac{2.7}{1.7} \approx 1.6$$

$c = 1 \cdot 6 \in (1, e)$ since $e \approx 2.7$. Thus Cauchy's mean value theorem is verified.

47. Show that the value of 'c' of Cauchy's mean value theorem for the functions sin x and cos x in (a, b) is the arithmetic mean between a and b (0 < a < b)

>> We have Cauchy's mean value theorem,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Let $\quad f(x) = \sin x \;;\; g(x) = \cos x$

$\therefore \quad f'(x) = \cos x \;;\; g'(x) = -\sin x$

$f(x)$ and $g(x)$ are continuous in $[a, b]$, differentiable in (a, b)
$g'(x) \neq 0 \ \ \forall \ x \in (a, b)$ since $0 < a < b$

Hence the theorem becomes,

$$\frac{\sin b - \sin a}{\cos b - \cos a} = \frac{\cos c}{-\sin c}$$

ie., $\quad -\sin b \sin c + \sin c \sin a = \cos b \cos c - \cos c \cos a$

or $\quad \cos c \cos a + \sin c \sin a = \cos b \cos c + \sin b \sin c$

ie., $\quad \cos(c-a) = \cos(b-c) \Rightarrow c - a = b - c$ or $2c = a + b$

$\therefore \quad$ **$c = (a+b)/2$ is the arithmetic mean of a and b. $c \in (a, b)$**

Note : *In order to find c we can also use the transformation formulae. The simplification is as follows.*

$$\text{From (1)} \quad -\cot c = \frac{2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{b+a}{2}\right)}{-2\sin\left(\frac{b-a}{2}\right)\sin\left(\frac{b+a}{2}\right)} = -\cot\left(\frac{b+a}{2}\right)$$

Thus $c = (b+a)/2$

---

48. *Show that the constant c of Cauchy's mean value theorem for the functions $1/x^2$ and $1/x$ in the interval [a, b] is the harmonic mean between a and b (0 < a < b)*

>> We have Cauchy's mean value theorem,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Let $\quad f(x) = 1/x^2 \; ; \; g(x) = 1/x$

$\therefore \quad f'(x) = -2/x^3, \quad g'(x) = -1/x^2$

$f(x)$ and $g(x)$ are continuous in $[a, b]$, differentiable in (a, b) and $g'(x) \neq 0$ $\forall \; x \in (a, b)$.

Hence the theorem becomes,

$$\frac{1/b^2 - 1/a^2}{1/b - 1/a} = \frac{-2/c^3}{-1/c^2} \quad \text{or} \quad \frac{a^2 - b^2/a^2 b^2}{a - b/ab} = \frac{2}{c}$$

ie.,
$$\frac{(a-b)(a+b)ab}{(a-b)\,a^2 b^2} = \frac{2}{c} \quad \text{or} \quad \frac{a+b}{ab} = \frac{2}{c}$$

Thus $\quad c = \dfrac{2ab}{a+b}$ **is the harmonic mean between a and** b . $c \in (a, b)$

Note on establishing inequalities by applying mean value theorems.

Suppose we need to establish an inequality of the form : $f_1(x) > f_2(x) > f_3(x)$ *where* $x > 0$, *we take* $F(x) = f_1(x) - f_2(x)$; $G(x) = f_2(x) - f_3(x)$ *and apply Lagrange's mean value theorem for* $F(x)$ *and also for* $G(x)$ *in the interval* $[0, x]$

---

49. *Apply mean value theorem to show that* $x > \log(1+x) > x - (1/2)x^2$ *where* $x > 0$

>> **Case-(i)** Let $f(x) = x - \log(1+x)$

$$\therefore \quad f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \; ; \text{ since } x > 0,\ f'(x) > 0$$

$f(x)$ is continuous in $[0, x]$ differentiable in $(0, x)$
Applying Lagrange's mean value theorem for this $f(x)$ we have

$$f(x) = f(0) + (x - 0)f'(c) \quad \text{But } f(0) = 0$$

$$\therefore \quad f(x) = xf'(c). \text{ Also } f'(c) > 0 \text{ since } c \in (0, x)$$

Hence $f(x) > 0$. That is $x - \log(1+x) > 0$

$$\therefore \quad x > \log(1+x) \qquad \ldots (1)$$

Case-(ii) Let $f(x) = \log(1+x) - x + (x^2/2)$

$$\therefore \quad f'(x) = \frac{1}{1+x} - 1 + x = \frac{1 - 1 - x + x + x^2}{1+x} = \frac{x^2}{1+x}$$

Clearly $f'(x) > 0$. Again applying the theorem in $[0, x]$ we have $f(x) = f(0) + xf'(c)$ But $f(0) = 0$

Also $f'(x) > 0 \quad \Rightarrow \quad f'(c) > 0$ since $c \in (0, x)$

Hence we have again $f(x) > 0$

ie., $\quad \log(1+x) - x + (x^2/2) > 0$

$$\therefore \quad \log(1+x) > x - (x^2/2) \qquad \ldots (2)$$

Thus by combining (1) and (2) we have

$$\mathbf{x > \log(1+x) > x - (x^2/2)}$$

---

50. *Show that for* $x > 0$, $\frac{x}{1+x} < \log(1+x) < x$

>> We shall establish the equivalent form of the desired result,

$$x > \log(1+x) > \frac{x}{1+x}$$

$x > \log(1+x)$ is same as Case-(i) of the previous problem.

Now, let $f(x) = \log(1+x) - \dfrac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \left\{ \frac{(1+x)\cdot 1 - x \cdot 1}{(1+x)^2} \right\} = \frac{1}{1+x} - \frac{1}{(1+x)^2}$$

$$f'(x) = \frac{(1+x)-1}{(1+x)^2} = \frac{x}{(1+x)^2}$$ Clearly $f'(x) > 0$ since $x > 0$.

Also $f(x)$ is continuous in $[0, x]$ and differentiable in $(0, x)$.

Applying Lagrange's mean value theorem for this $f(x)$ in $[0, x]$ we have,

$$f(x) = f(0) + (x-0)f'(c) \quad \text{But } f(0) = 0$$

$\therefore \quad f(x) = xf'(c)$; $f'(x) > 0 \Rightarrow f'(c) > 0$ and hence $f(x) > 0$

ie., $\quad \log(1+x) - \frac{x}{1+x} > 0$ or $\log(1+x) > \frac{x}{1+x}$

Since we also have $x > \log(1+x)$, combining these we get

$$\mathbf{x > \log(1+x) > \frac{x}{1+x} \quad \text{or} \quad \frac{x}{1+x} < \log(1+x) < x}$$

---

51. *Prove that* $\cos x > 1 - (x^2/2)$ *for* $x > 0$

>> Let $f(x) = \cos x - 1 + (x^2/2)$

$$f'(x) = -\sin x + x$$

Since $x > 0$, $\sin x < x$ or $x - \sin x > 0 \quad \therefore \quad f'(x) > 0$

The function $f(x)$ is continuous in $[0, x]$, differentiable in $(0, x)$.
Hence by applying Lagrange's mean value theorem for $f(x)$ in $[0, x]$ we have,

$$f(x) = f(0) + (x-0)f'(c)$$

But $\quad f(0) = 0 \;\therefore\; f(x) = xf'(c)$. $f'(x) > 0 \Rightarrow f'(c) > 0$

Hence $f(x) > 0$. That is, $\cos x - 1 + (x^2/2) > 0$

Thus $\quad \mathbf{\cos x > 1 - (x^2/2)}$

---

52. *Prove that* $\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$ *where* $a < b < 1$

>> Let $f(x) = \sin^{-1} x \quad \therefore \quad f'(x) = \frac{1}{\sqrt{1-x^2}}$

$f(x)$ is continuous in $[a, b]$ and differentiable in (a, b).
Applying Lagrange's mean value theorem for $f(x)$ in $[a, b]$ we get when $a < c < b$

$$\frac{\sin^{-1} b - \sin^{-1} a}{b - a} = \frac{1}{\sqrt{1-c^2}} \qquad \ldots (1)$$

Now $a < c \Rightarrow a^2 < c^2 \Rightarrow -a^2 > -c^2 \Rightarrow 1 - a^2 > 1 - c^2$

Hence $\dfrac{1}{\sqrt{1-a^2}} < \dfrac{1}{\sqrt{1-c^2}}$ . . . (2)

Also $c < b$. On similar lines, $\dfrac{1}{\sqrt{1-c^2}} < \dfrac{1}{\sqrt{1-b^2}}$ . . . (3)

Combining (2) and (3) we get,

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

or $$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$ by using (1).

On multiplying by $(b-a)$ which is positive, we have

$$\mathbf{\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}}$$

Note : [illegible]

(*We need to take* $f(x) = \tan^{-1} x$)

---

53. *Prove that* [illegible]

>> Let $f(x) = \sin x$. By applying Lagrange's mean value theorem for $f(x)$ in $(x, x+h)$ we get,

$$\frac{\sin(x+h) - \sin x}{x+h-x} = f'(c) \quad ; \text{But } f'(x) = \cos x$$

$$\therefore \quad \frac{\sin(x+h) - \sin x}{h} = \cos c$$

Thus $\mathbf{\sin(x+h) - \sin x = h \cos c}$ as required.

---

54. *Prove that* [illegible]

>> Let $f(x) = \sin x$ and $g(x) = \cos x$

$\therefore \quad f'(x) = \cos x$ and $g'(x) = -\sin x$

Applying Cauchy's mean value theorem we have,

$$\frac{\sin b - \sin a}{\cos b - \cos a} = \frac{\cos c}{-\sin c} \quad \text{or} \quad \frac{\sin b - \sin a}{\cos b - \cos a} = -\cot c$$

Thus $\dfrac{\sin b - \sin a}{\cos a - \cos b} = \cot c$ as required.

EXERCISES

Verify Rolle's theorem for the following functions.

1. $f(x) = \sin x / e^x$ in $[0, \pi]$
2. $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$
3. $f(x) = (x-a)[(x-a)(x-b)]^2$ in $[a, b]$

Verify Lagrange's mean value theorem for the following functions.

4. $f(x) = \log x$ in $[e, e^2]$
5. $f(x) = (x-2)(x+2)(x-3)$ in $[1, 4]$
6. $f(x) = px^2 + qx + r$ in $[a, b]$

Verify Cauchy's mean value theorem for the following pairs of functions.

7. $\sin x$ and $\cos x$ in $[\pi/4, 3\pi/4]$
8. e^x and $1/e^x$ in $[a, b]$
9. $f(x)$ and $2f'(x)$ in $[a, b]$, where $f(x) = \sqrt{x}$
10. Show that if $0 < a < b$,

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

and hence deduce that

$$\frac{5\pi + 4}{20} < (\tan^{-1} 2) < \frac{\pi + 2}{4}$$

ANSWERS

1. $c = \pi/4$ 2. $c = -2$ 3. $c = (2a + 3b)/5$

4. $c = e^2 - e$ 5. $c = 1 + \sqrt{3}$ 6. $c = (a + b)/2$

7. $c = \pi$ 8. $c = (a + b)/2$ 9. $c = \sqrt{ab}$

1.3 Expansion of functions

We have Taylor's theorem in the form

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h)$$

Evidently the expression in the R.H.S contains $(n+1)$ terms and we denote $R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h)$ which is called the remainder after n terms.

Let $a+h = x$ or $h = (x-a)$. If x is close enough to 'a' then h will be very small and $R_n \to 0$ as $n \to \infty$. As $n \to \infty$ the number of terms increase indefinitely and we have an infinite series expansion of $f(x)$ in powers of $(x-a)$ given by

$$\boxed{f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)+\ldots}$$

This is called ***Taylor's series expansion of*** $f(x)$ about the point 'a'.

In particular if $a = 0$ we have,

$$\boxed{f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots}$$

This is called ***Maclaurin's series expansion*** of $f(x)$

Working procedure for problems

For convenience we shall use the notation

$y(x)$ for $f(x)$ and $y_1(x)$, $y_2(x)$, $y_3(x)$ ... respectively for

$f'(x)$, $f''(x)$, $f'''(x)$, ... so that we have

$$y(x) = y(a)+(x-a)y_1(a)+\frac{(x-a)^2}{2!}y_2(a)+\ldots \qquad \text{[\textit{Taylor's expansion}]}$$

$$y(x) = y(0)+xy_1(0)+\frac{x^2}{2!}y_2(0)+\ldots \qquad \text{[\textit{Maclaurin's expansion}]}$$

- We need to find successive derivatives of the given $y(x)$ and evaluate them at the given point $x = a$ for obtaining the Taylor's expansion and evaluate at $x = 0$ for obtaining the Maclaurin's expansion.
- To reduce the computational work we must prefer to use indirect methods for obtaining various derivatives of the given function which we are familiar in the discussion of the topic '***Successive differentiation***'

WORKED PROBLEMS

55. *Obtain the Taylor's expansion of* $\log_e x$ *about* $x = 1$ *upto the term containing fourth degree and hence obtain* $\log_e (1.1)$

\>> We have Taylor's expansion about $x = a$ given by

$$y(x) = y(a) + (x-a)\,y_1(a) + \frac{(x-a)^2}{2!}\,y_2(a) + \ldots$$

By data, $y(x) = \log_e x$; $a = 1$

$y(1) = \log_e 1 = 0$. Differentiating $y(x)$ successively we get,

$$y_1(x) = \frac{1}{x} \quad \therefore \; y_1(1) = 1 \quad ; \quad y_3(x) = \frac{2}{x^3} \quad \therefore \; y_3(1) = 2$$

$$y_2(x) = -\frac{1}{x^2} \quad \therefore \; y_2(1) = -1 \quad ; \quad y_4(x) = -\frac{6}{x^4} \quad \therefore \; y_4(1) = -6$$

Taylor's series upto fourth degree term with $a = 1$ is given by

$$y(x) = y(1) + (x-1)\,y_1(1) + \frac{(x-1)^2}{2!}\,y_2(1) + \frac{(x-1)^3}{3!}\,y_3(1) + \frac{(x-1)^4}{4!}\,y_4(1)$$

Hence, $\log_e x = 0 + (x-1)\,1 + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(-6)$

Thus $\boldsymbol{\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}}$

Now putting $x = 1.1$ we have

$$\log_e(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = \mathbf{0.0953}$$

---

56. *Expand* $\tan^{-1} x$ *in powers of* $(x-1)$ *upto the term containing fourth degree.*

\>> Taylor's expansion in powers of $(x-1)$ is given by

$$y(x) = y(1) + (x-1)\,y_1(1) + \frac{(x-1)^2}{2!}\,y_2(1) + \frac{(x-1)^3}{3!}(1) + \frac{(x-1)^4}{4!}\,y_4(1) + \ldots$$

$y = \tan^{-1} x \quad \therefore \; y(1) = \tan^{-1} 1 = \pi/4$

$y_1 = \frac{1}{1+x^2} \quad \therefore \; y_1(1) = \frac{1}{2}$, (*We donot prefer direct differentiation*)

We have $(1+x^2)\,y_1 = 1$ ... (1)

We have to successively differentiate to obtain expressions involving y_2, y_3, y_4 and [them] at $x = 1$

[Hen]ce we have on differentiating (1),

$$(1+x^2)\,y_2 + 2xy_1 = 0 \quad \ldots (2)$$

Putting $x = 1$; $2y_2(1) + 2\cdot 1\cdot \frac{1}{2} = 0 \quad \therefore \quad y_2(1) = -1/2$

Differentiating (2) w.r.t x, we get,

$$(1+x^2)\,y_3 + 4xy_2 + 2y_1 = 0 \quad \ldots (3)$$

Putting $x = 1$; $2y_3(1) - 2 + 1 = 0 \quad \therefore \quad y_3(1) = 1/2$

Differentiating (3) w.r.t x, we get,

$$(1+x^2)\,y_4 + 6xy_3 + 6y_2 = 0 \quad \ldots (4)$$

Putting $x = 1$; $2y_4(1) + 3 - 3 = 0 \quad \therefore \quad y_4(1) = 0$

Substituting these values in the expansion we get

$$\tan^{-1}x = \frac{\pi}{4} + \frac{1}{2}\left\{ (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} \right\}$$

57. [illegible]

\>> We have by Taylor's theorem,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots$$

Taking $a = \pi/4$, $f(\pi/4+h) = \sin(\pi/4+h) \Rightarrow f(x) = \sin x$

or $y(x) = \sin x$

$$\therefore \quad y(\pi/4+h) = y(\pi/4) + h\,y_1(\pi/4) + \frac{h^2}{2!}y_2(\pi/4) + \frac{h^3}{3!}y_3(\pi/4) + \frac{h^4}{4!}y_4(\pi/4) \quad \ldots (1)$$

Consider $y(x) = \sin x \quad \therefore \quad y(\pi/4) = 1/\sqrt{2}$

$y_1(x) = \cos x \quad \therefore \quad y_1(\pi/4) = 1/\sqrt{2}$

$y_2(x) = -\sin x \quad \therefore \quad y_2(\pi/4) = -1/\sqrt{2}$

$y_3(x) = -\cos x \quad \therefore \quad y_3(\pi/4) = -1/\sqrt{2}$

$y_4(x) = \sin x \qquad \therefore \qquad y_4(\pi/4) = 1/\sqrt{2}$

Substituting these values in (1) we obtain

$$\sin(\pi/4 + h) = \frac{1}{\sqrt{2}}\left\{1 + h - \frac{h^2}{2!} - \frac{h^3}{3!} + \frac{h^4}{4!}\right\} \qquad \ldots (2)$$

To find $\sin 50^\circ$ we have to take $h = 5^\circ$

ie., $h = 5 \cdot (\pi/180)$ radians $= \pi/36 \approx 0.087$

Substituting $h = 0.087$ in the R.H.S of (2) we obtain,

$$\sin 50^\circ = 0.7659$$

[illegible]

>> Taylor's expansion of $y(x)$ about $x = \pi/3$ is given by

$$y(x) = y(\pi/3) + (x - \pi/3)\, y_1(\pi/3) + \frac{(x-\pi/3)^2}{2!}\, y_2(\pi/3) + \frac{(x-\pi/3)^3}{3!}\, y_3(\pi/3) + \frac{(x-\pi/3)^4}{4!}\, y_4(\pi/3) + \cdots \qquad \ldots (1)$$

Let $y(x) = \log(\cos x)$

$\therefore \quad y(\pi/3) = \log[\cos(\pi/3)] = \log(1/2) = -\log 2$

$y_1 = \frac{1}{\cos x} \cdot -\sin x$

ie., $y_1 = -\tan x \qquad \therefore \quad y_1(\pi/3) = -\tan(\pi/3) = -\sqrt{3}$

$y_2 = -\sec^2 x = -(1 + \tan^2 x)$

ie., $y_2 = -(1 + y_1^2) \qquad \therefore \quad y_2(\pi/3) = -[1 + (\sqrt{3})^2] = -4$

$y_3 = -2\, y_1 y_2 \qquad \therefore \quad y_3(\pi/3) = -2 \cdot -\sqrt{3} \cdot -4 = -8\sqrt{3}$

$y_4 = -2[y_1 y_3 + y_2^2] \qquad \therefore \quad y_4(\pi/3) = -2[-\sqrt{3} \cdot -8\sqrt{3} + 16] = -80$

Substituting these values in (1) we have,

$$\log(\cos x) = -\log 2 - (x - \pi/3)\sqrt{3} - \frac{(x-\pi/3)^2}{2} \cdot 4 - \frac{(x-\pi/3)^3}{6} \cdot 8\sqrt{3} - \frac{(x-\pi/3)^4}{24} \cdot 80 + \cdots$$

Thus $\log(\cos x) = -\log 2 - \sqrt{3}\,(x - \pi/3) - 2\,(x - \pi/3)^2$
$$- \frac{4}{\sqrt{3}}(x - \pi/3)^3 - \frac{10}{3}(x - \pi/3)^4$$

---

59. Obtain the Maclaurin's [illegible]

>> We have $y(x) = y(0) + x\,y_1(0) + \dfrac{x^2}{2!}\,y_2(0) + \ldots$

$y = \sin^{-1} x \qquad \therefore \quad y(0) = \sin^{-1} 0 = 0$.

$$y_1 = \frac{1}{\sqrt{1 - x^2}} \qquad \ldots (1)$$

[*Finding* y_2, y_3, y_4, y_5 *becomes very difficult through direct differentiation. Hence we shall adopt the technique of avoiding denominator by cross multiplying and squaring for avoiding square root*]. We have $y_1(0) = 1$.

Thus we have $(1 - x^2)\,y_1^2 = 1$

Differentiating now w.r.t. x, we get,

$(1 - x^2)\,2\,y_1\,y_2 + y_1^2\,(-2x) = 0$. Dividing by $2y_1$ we get,

$$(1 - x^2)\,y_2 - x y_1 = 0 \qquad \ldots (2)$$

Now putting $x = 0$ we have $1 \cdot y_2(0) - 0 = 0 \qquad \therefore \quad y_2(0) = 0$.

[*Here* y_1, y_2 *is to be understood as* $y_1(x)$, $y_2(x)$].

Now differentiating (2) w.r.t. x we obtain

$(1 - x^2)\,y_3 + y_2\,(-2x) - [x y_2 + y_1 \cdot 1] = 0$

ie., $$(1 - x^2)\,y_3 - 3x y_2 - y_1 = 0 \qquad \ldots (3)$$

Putting $x = 0$ we have $y_3(0) - 0 - 1 = 0, \qquad \therefore \quad y_3(0) = 1$

Differentiating (3) again w.r.t. x we obtain,

$(1 - x^2)\,y_4 + y_3\,(-2x) - 3\,[x y_3 + y_2 \cdot 1] - y_2 = 0$.

ie., $$(1 - x^2)\,y_4 - 5x y_3 - 4 y_2 = 0 \qquad \ldots (4)$$

Putting $x = 0$ we have $y_4(0) - 0 - 0 = 0 \qquad \therefore \quad y_4(0) = 0$

Differentiating (4) again w.r.t. x we have,

$$(1-x^2)\,y_5 + y_4(-2x) - 5[\,xy_4 + y_3 \cdot 1\,] - 4y_3 = 0$$

ie., $$(1-x^2)\,y_5 - 7xy_4 - 9y_3 = 0 \qquad \ldots (5)$$

Putting $x = 0$ we have $y_5(0) - 0 - 9 \cdot 1 = 0 \quad \therefore \quad y_5(0) = 9$

Substituting these values in the expansion of $y(x)$ we have,

$$\sin^{-1} x = 0 + x \cdot 1 + \frac{x^2}{2} \cdot 0 + \frac{x^3}{6} \cdot 1 + \frac{x^4}{24} \cdot 0 + \frac{x^5}{120} \cdot 9$$

Thus $\mathbf{\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40}}$

Note [illegible]

Expand $\sin^{-1} x$ *in ascending powers of x upto the first three non vanishing terms.*

[illegible]

>> We have Maclaurin's expansion,

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \frac{x^3}{3!}\,y_3(0) + \frac{x^4}{4!}\,y_4(0) + \ldots$$

Let $y = e^{\sin x} \qquad \therefore \quad y(0) = e^0 = 1$

$y_1 = e^{\sin x} \cos x$ or $y_1 = y \cos x \quad \therefore \quad y_1(0) = y(0) \cdot \cos 0 = 1$

$y_2 = -y \sin x + \cos x \cdot y_1 \qquad \therefore \quad y_2(0) = 0 + 1 = 1$

$y_3 = -(y \cos x + y_1 \sin x) + (\cos x\, y_2 - y_1 \sin x)$

$= -y_1 - 2y_1 \sin x + \cos x \cdot y_2 \qquad \therefore \quad y_3(0) = -1 - 0 + 1 = 0$

$y_4 = -y_2 - 2(y_1 \cos x + \sin x\;\, y_2) + (\cos x\, y_3 - \sin x\;\, y_2)$

$= -y_2 - 2\,y_1 \cos x - 3 \sin x\, y_2 + \cos x \cdot y_3$

$y_4(0) = -1 - 2 - 0 + 0 = -3 \qquad \therefore \quad y_4(0) = -3$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

61. *Expand* log (1 + sin x) [illegible] Maclaurin's theorem upto the term containing x^4

>> We have Maclaurin's expansion

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \ldots$$

Consider $y = \log(1 + \sin x)$ $\therefore \quad y(0) = \log 1 = 0$

$$y_1 = \frac{\cos x}{1 + \sin x} \qquad \therefore \quad y_1(0) = 1$$

ie., $(1 + \sin x)\, y_1 = \cos x$ . . . (1)

Differentiating w.r.t. x we get,

$$(1 + \sin x)\, y_2 + \cos x \; y_1 = -\sin x \qquad \ldots (2)$$

At $x = 0$ we get $y_2(0) + 1 = 0$ $\therefore \quad y_2(0) = -1$

Differentiating (2) again we get,

$$(1 + \sin x)\, y_3 + 2y_2 \cos x - y_1 \sin x = -\cos x \qquad \ldots (3)$$

At $x = 0$ we get, $y_3(0) - 2 - 0 = -1$ $\therefore \quad y_3(0) = 1$.

Differentiating (3) again we get,

$$(1 + \sin x)\, y_4 + \cos x \; y_3 + 2(-y_2 \sin x + \cos x \; y_3) - (y_1 \cos x + \sin x \; y_2) = \sin x$$

ie., $(1 + \sin x)\, y_4 + 3\cos x\, y_3 - 3y_2 \sin x - y_1 \cos x = \sin x$

At $x = 0$ we get, $y_4(0) + 3 - 0 - 1 = 0$ $\therefore \quad y_4(0) = -2$

Thus by substituting these values in the expansion we get

$$\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}$$

62. [illegible]

Expand log (sec x) [illegible]

>> $y(x) = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \ldots$

$y = \log(\sec x)$ $\therefore \quad y(0) = \log 1 = 0$

$$y_1 = \frac{\sec x \tan x}{\sec x} \quad ie., \; y_1 = \tan x \quad \therefore \; y_1(0) = 0$$

$$y_2 = \sec^2 x \quad \therefore \; y_2(0) = 1.$$

Now $y_2 = 1 + \tan^2 x = 1 + y_1^2$

Differentiating this w.r.t. x successively we have,

$$y_3 = 2y_1 y_2 \qquad \therefore \; y_3(0) = 0$$

$$y_4 = 2(y_1 y_3 + y_2^2) \qquad \therefore \; y_4(0) = 2$$

$$y_5 = 2(y_1 y_4 + y_2 y_3 + 2y_2 y_3) = 2y_1 y_4 + 6y_2 y_3 \qquad \therefore \; y_5(0) = 0$$

$$y_6 = 2(y_1 y_5 + y_2 y_4) + 6(y_2 y_4 + y_3^2)$$

$$ie., \quad y_6 = 2y_1 y_5 + 8y_2 y_4 + 6y_3^2 \qquad \therefore \; y_6(0) = 16$$

Substituting these values in the expansion of $y(x)$ we get,

$$\log(\sec x) = \frac{x^2}{2}\cdot 1 + \frac{x^4}{24}\cdot 2 + \frac{x^6}{720}\cdot 16$$

Thus $\mathbf{\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}}$

>> $$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \frac{x^3}{3!}\,y_3(0) + \cdots$$

Consider $y = \tan^{-1}(1+x)$ $\qquad \therefore \; y(0) = \tan^{-1}(1) = \pi/4$

$$y_1 = \frac{1}{1 + (1+x)^2} = \frac{1}{x^2 + 2x + 2} \qquad \therefore \; y_1(0) = 1/2$$

$$ie., \quad (x^2 + 2x + 2)\,y_1 = 1 \qquad \ldots (1)$$

Differentiating w.r.t. x,

$$(x^2 + 2x + 2)\,y_2 + 2(x+1)\,y_1 = 0 \qquad \ldots (2)$$

At $x = 0$ we get $2y_2(0) + 2\cdot(1/2) = 0$ $\qquad \therefore \; y_2(0) = -1/2$

Differentiating (2) w.r.t. x we have,

$$(x^2 + 2x + 2)\,y_3 + 2(x+1)\,y_2 + 2(x+1)\,y_2 + 2y_1 = 0$$

ie., $(x^2+2x+2)\,y_3+4\,(x+1)\,y_2+2y_1 = 0$ . . . (3)

At $x = 0$ we get, $2y_3(0)+4\cdot(-1/2)+2\cdot(1/2) = 0 \quad \therefore \quad y_3(0) = 1/2$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$\tan^{-1}(1+x) = \frac{\pi}{4}+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{12}$$

---

64. *Using Maclaurin's series prove that* $\sqrt{1+\sin 2x} = 1+x-\dfrac{x^2}{2}-\dfrac{x^3}{6}+\dfrac{x^4}{24}$

>> We have, $y(x) = y(0)+x\,y_1(0)+\dfrac{x^2}{2!}y_2(0)+\ldots$

Let $y = \sqrt{1+\sin 2x} = \sqrt{\cos^2 x+\sin^2 x+2\sin x\cos x}$

$= \sqrt{(\cos x+\sin x)^2} = \cos x+\sin x$

Thus $y = \cos x+\sin x \qquad \therefore \quad y(0) = 1$

$y_1 = -\sin x+\cos x \qquad \therefore \quad y_1(0) = 1$

$y_2 = -\cos x-\sin x = -y \quad$ *ie.,* $y_2 = -y \qquad \therefore \quad y_2(0) = -1$

$\therefore \quad y_3 = -y_1\ ;\ y_3(0) = -1.\quad y_4 = -y_2 \qquad \therefore \quad y_4(0) = 1$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$\sqrt{1+\sin 2x} = 1+x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}\ \ldots$$

Remark : *We have used a technique to simplify the given function leading to a very simple form, there by the problem is completed easily. The usual procedure of squaring $y(x)$ and differentiating thereon can also be done.*

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65. *Obtain the Maclaurin's expansion of the function* $\log(1+x)$ *and hence deduce that* $\log\sqrt{\dfrac{1+x}{1-x}} = x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\ldots$

>> We have $y(x) = y(0)+x\,y_1(0)+\dfrac{x^2}{2!}y_2(0)+\dfrac{x^3}{3!}y_3(0)+\ldots$

Let $y = \log(1+x) \quad \therefore \quad y(0) = \log 1 = 0$

$y_1 = \dfrac{1}{1+x} \qquad \therefore \quad y_1(0) = 1\ ;\quad y_2 = \dfrac{-1}{(1+x)^2} \qquad \therefore \quad y_2(0) = -1$

$y_3 = \dfrac{2}{(1+x)^3} \qquad \therefore \quad y_3(0) = 2\ ;\quad y_4 = \dfrac{-6}{(1+x)^4} \qquad \therefore \quad y_4(0) = -6$

$$y_5 = \frac{24}{(1+x)^5} \quad \therefore \quad y_5(0) = 24$$

Substituting these values in the expansion we get,

$$\log(1+x) = 0 + x \cdot 1 + \frac{x^2}{2} \cdot (-1) + \frac{x^3}{6} \cdot 2 + \frac{x^4}{24} \cdot (-6) + \frac{x^5}{120} \cdot 24 - \ldots$$

Thus $\mathbf{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots}$ ... (1)

Next, $\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left\{\log(1+x) - \log(1-x)\right\}$ ... (2)

Changing x to $-x$ in (1) we obtain

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots \qquad \ldots (3)$$

Using (1) and (3) in (2) we have

$$\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left\{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \ldots\right)\right\}$$

ie., $\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left(2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \ldots\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$

Thus $\mathbf{\log\sqrt{1+x/1-x} = x + (x^3/3) + (x^5/5) + \cdots}$

[illegible]

>> $y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0)$

Let $y = \tan(\pi/4 + x) \qquad \therefore \quad y(0) = \tan(\pi/4) = 1$

$y_1 = \sec^2(\pi/4 + x) = 1 + y^2 \qquad \therefore \quad y_1(0) = 2$

$y_2 = 2yy_1 \qquad \therefore \quad y_2(0) = 4$

$y_3 = 2(yy_2 + y_1^2) \qquad \therefore \quad y_3(0) = 2(4+4) = 16$

$y_4 = 2(y\,y_3 + 3\,y_1\,y_2) \qquad \therefore \quad y_4(0) = 2(16 + 24) = 80$

Substituting these values in the expansion of $y(x)$ we have,

$$\tan(\pi/4 + x) = 1 + x \cdot 2 + \frac{x^2}{2} \cdot 4 + \frac{x^3}{6} \cdot 16 + \frac{x^4}{24} \cdot 80$$

Thus $\mathbf{\tan(\pi/4 + x) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4}$

---

67. [illegible]

>> $$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \frac{x^3}{3!}\,y_3(0) + \ldots$$

Let $y = \log\tan(\pi/4 + x)$; $y(0) = \log 1 = 0$

Also $e^y = \tan(\pi/4 + x)$

Differentiating w.r.t. x we get,

$$e^y\,y_1 = \sec^2(\pi/4 + x) = 1 + (e^y)^2$$

ie., $e^y\,y_1 = 1 + e^{2y}$ . . . (1)

At $x = 0,\ 1 \cdot y_1(0) = 2$ $\therefore\ y_1(0) = 2$

Differentiating (1) w.r.t. x we get,

$$e^y\,y_2 + e^y\,y_1^2 = 2\,e^{2y}\,y_1 \quad \text{or} \quad y_2 + y_1^2 = 2\,e^y\,y_1 \qquad \ldots (2)$$

At $x = 0$, $y_2(0) + 4 = 4$ $\therefore\ y_2(0) = 0$

Differentiating (2) w.r.t. x we get,

$$y_3 + 2\,y_1\,y_2 = 2\,(e^y\,y_2 + e^y\,y_1^2) \qquad \ldots (3)$$

At $x = 0$, $y_3(0) + 0 = 2(0 + 4)$ $\therefore\ y_3(0) = 8$

Differentiating (3) w.r.t. x we get,

$$y_4 + 2\,y_1\,y_3 + 2\,y_2^2 = 2\,(e^y\,y_3 + e^y\,y_1\,y_2 + 2\,e^y\,y_1\,y_2 + e^y\,y_1^3)$$

ie., $$y_4 + 2\,y_1\,y_3 + 2\,y_2^2 = 2\,e^y\,(y_3 + 3\,y_1\,y_2 + y_1^3) \qquad \ldots (4)$$

At $x = 0$, $y_4(0) + 2 \times 2 \times 8 + 0 = 2(8 + 0 + 8)$ $\therefore\ y_4(0) = 0$

Differentiating (4) w.r.t. x we get,

$$y_5 + 2\,y_1\,y_4 + 2\,y_2\,y_3 + 4\,y_2\,y_3 = 2\,e^y\,(y_4 + 3\,y_1\,y_3 + 3\,y_2^2 + 3\,y_1^2\,y_2) + 2\,e^y\,y_1\,(y_3 + 3\,y_1\,y_2 + y_1^3)$$

At $x = 0$, $y_5(0) = 2(48) + 2 \cdot 2(8 + 8) = 160$

Substituting these values in the expansion of $y(x)$ we have,

$$\log\tan(\pi/4 + x) = x \cdot 2 + \frac{x^3}{6} \cdot 8 + \frac{x^5}{120} \cdot 160 + \ldots$$

Thus $$\boldsymbol{\log\tan(\pi/4 + x) = 2x + \frac{4x^3}{3} + \frac{4x^5}{3} + \ldots}$$

68. *Expand $e^{x \sin x}$ in ascending powers of x upto the fourth degree terms.*

>> $y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \ldots$

Let $y = e^{x \sin x}$ $\qquad \therefore \quad y(0) = 1$

$y_1 = e^{x \sin x}(x \cos x + \sin x)$ $\qquad \therefore \quad y_1(0) = 0$

ie., $y_1 = y(x \cos x + \sin x)$

Now $y_2 = y(-x \sin x + 2 \cos x) + y_1(x \cos x + \sin x)$

Hence $y_2(0) = 1(0 + 2) + 0$ $\qquad \therefore \quad y_2(0) = 2$

Next, $y_3 = y(-x \cos x - 3 \sin x) + y_1(-x \sin x + 2 \cos x)$

$+ y_1(-x \sin x + 2 \cos x) + y_2(x \cos x + \sin x)$

ie., $y_3 = y(-x \cos x - 3 \sin x) + 2y_1(-x \sin x + 2 \cos x) + y_2(x \cos x + \sin x)$

Hence $y_3(0) = 0 + 0 + 0$ $\qquad \therefore \quad y_3(0) = 0$

Next $y_4 = y(x \sin x - 4 \cos x) + y_1(-x \cos x - 3 \sin x)$

$+ 2y_1(-x \cos x - 3 \sin x) + 2y_2(-x \sin x + 2 \cos x)$

$+ y_2(-x \sin x + 2 \cos x) + y_3(x \cos x + \sin x)$

Hence $y_4(0) = -4 + 0 + 0 + 8 + 4 = 8$ $\qquad \therefore \quad y_4(0) = 8$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$e^{x \sin x} = 1 + x^2 + (x^4/3) + \ldots$$

---

69. *Obtain the Maclaurin's expansion of $\log(1 + e^x)$ as far as the fourth degree terms.*

>> $y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \ldots$

Let $y = \log(1 + e^x)$ $\qquad \therefore \quad y(0) = \log_e 2$

$y_1 = \dfrac{e^x}{1 + e^x}$ $\qquad \therefore \quad y_1(0) = \dfrac{1}{2}$

ie., $(1 + e^x)\,y_1 = e^x$ $\qquad \ldots (1)$

Differentiating w.r.t. x we get,

$$(1+e^x)\,y_2 + e^x\,y_1 = e^x \qquad \ldots (2)$$

At $x = 0$, $2\,y_2(0) + 1/2 = 1$ $\therefore$ $y_2(0) = 1/4$

Differentiating (2) w.r.t. x we get,

$$(1+e^x)\,y_3 + 2\,e^x\,y_2 + e^x\,y_1 = e^x \qquad \ldots (3)$$

At $x = 0$, $2y_3(0) + 1/2 + 1/2 = 1$ $\therefore$ $y_3(0) = 0$

Differentiating (3) w.r.t. x we get,

$$(1+e^x)\,y_4 + 3e^x\,y_3 + 3e^x\,y_2 + e^x\,y_1 = e^x \qquad \ldots (4)$$

At $x = 0$, $2y_4(0) + 3/4 + 1/2 = 1$ $\therefore$ $y_4(0) = -1/8$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$\log(1+e^x) = \log_e 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192}$$

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70. [illegible]

>> Maclaurin's series is given by

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \frac{x^3}{3!}\,y_3(0) + \frac{x^4}{4!}\,y_4(0) + \cdots$$

Consider $y = y(x) = \log(1+\cos x)$; $y(0) = \log_e 2$

$$y_1 = \frac{-\sin x}{1+\cos x} \qquad \therefore \quad y_1(0) = 0$$

ie., $(1+\cos x)\,y_1 = -\sin x$ $\ldots (1)$

Differentiating w.r.t x we have,

$$(1+\cos x)\,y_2 - \sin x\;y_1 = -\cos x \qquad \ldots (2)$$

At $x = 0$, we get $2y_2(0) - 0 = -1$ $\therefore$ $y_2(0) = -1/2$

Differentiating (2) again w.r.t x we have

$$(1+\cos x)\,y_3 - \sin x\;y_2 - [\sin x\;y_2 + \cos x\;y_1] = \sin x$$

ie., $(1+\cos x)\,y_3 - 2\sin x\;y_2 - \cos x\;y_1 = \sin x$ $\ldots (3)$

At $x = 0$, we get $2\,y_3(0) - 0 - 0 = 0$ $\therefore$ $y_3(0) = 0$

Differentiating (3) again w.r.t x we have,

$$(1+\cos x)\,y_4 - \sin x\ y_3 - 2\,[\sin x\ y_3 + \cos x\ y_2] - [\cos x\ y_2 - \sin x\ y_1] = \cos x$$

ie., $(1+\cos x)\,y_4 - 3\sin x\ y_3 - 3\cos x\ y_2 + \sin x\ y_1 = \cos x$ ... (4)

At $x = 0$, we get $2y_4(0) - 0 + 3/2 + 0 = 1 \quad \therefore \quad y_4(0) = -1/4$

Thus the required Maclaurin's series is given by

$$\log_e(1+\cos x) = \log_e 2 - \frac{x^2}{4} - \frac{x^4}{96} \dots$$

Aliter

$$y = \log(1+\cos x) = \log[2\cos^2(x/2)]$$

ie., $y = \log 2 + 2\log\cos(x/2) \quad \therefore \quad y(0) = \log_e 2$

Now $y_1 = -\tan(x/2) \quad \therefore \quad y_1(0) = 0$

$$y_2 = -\frac{1}{2}\sec^2(x/2) \quad \therefore \quad y_2(0) = -1/2$$

Also $y_2 = -\frac{1}{2}[1+\tan^2(x/2)] = -\frac{1}{2}(1+y_1^2)$

$\therefore \quad y_3 = -\frac{1}{2}(2y_1 y_2) = -y_1 y_2 \quad \therefore \quad y_3(0) = 0$

$y_4 = -y_1 y_3 - y_2^2 \quad \therefore \quad y_4(0) = -1/4$

Thus $\log(1+\cos x) = \log_e 2 - (x^2/4) - (x^4/96) \cdots$

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71. Obtain the Maclaurin's expansion of a^x

\>> $y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + ..$

Let $y = a^x \quad \therefore \quad y(0) = 1$

$y_1 = a^x \log a = y\log a \quad \therefore \quad y_1(0) = \log a$

$y_2 = y_1 \log a \quad \therefore \quad y_2(0) = (\log a)^2$

$y_3 = y_2 \log a \quad \therefore \quad y_3(0) = (\log a)^3$ and so on.

Substituting these values in the expansion of $y(x)$ we get,

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \dots$$

---

72. *Expand* $\tan^{-1} x$ *in ascending powers of x upto the first three non zero terms and hence show that* $\pi = 4(1 - 1/3 + 1/5 - \ldots)$

>> $$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots$$

Let $y = \tan^{-1} x \quad \therefore \quad y(0) = \tan^{-1}(0) = 0$

$y_1 = \dfrac{1}{1+x^2} \quad \therefore \quad y_1(0) = 1$

ie., $(1 + x^2)\,y_1 = 1$ . . . (1)

Differentiating w.r.t. x we get,

$(1 + x^2)\,y_2 + 2xy_1 = 0$ . . . (2)

At $x = 0$ we get, $y_2(0) = 0 \quad \therefore \quad y_2(0) = 0$

Differentiating (2) again we get $(1 + x^2)\,y_3 + 2x\,y_2 + 2x\,y_2 + 2y_1 = 0$

ie., $(1 + x^2)\,y_3 + 4xy_2 + 2y_1 = 0$ . . . (3)

At $x = 0$ we get, $y_3(0) = -2. \quad \therefore \quad y_3(0) = -2$

Differentiating (3) again w.r.t.x, we get

$(1 + x^2)\,y_4 + 2xy_3 + 4xy_3 + 4y_2 + 2y_2 = 0$

ie., $(1 + x^2)\,y_4 + 6xy_3 + 6y_2 = 0$ . . . (4)

At $x = 0$ we get, $y_4(0) = 0 \quad \therefore \quad y_4(0) = 0$

Differentiating (4) again w.r.t. x we get,

$(1 + x^2)\,y_5 + 2xy_4 + 6xy_4 + 6y_3 + 6y_3 = 0$

ie., $(1 + x^2)\,y_5 + 8xy_4 + 12y_3 = 0$ . . . (5)

At $x = 0$ we get, $y_5(0) = 24 \quad \therefore \quad y_5(0) = 24$

Substituting these values in the expansion we have

$$\tan^{-1} x = x \cdot 1 + \frac{x^3}{6} \cdot -2 + \frac{x^5}{120} \cdot 24 - \ldots$$

Thus $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Putting $x = 1$ we get $\tan^{-1} 1 = 1 - 1/3 + 1/5 - 1/7 + \ldots$

ie., $\pi/4 = 1 - 1/3 + 1/5 - \ldots$ Thus $\boldsymbol{\pi = 4(1 - 1/3 + 1/5 - \ldots)}$

Note - 1 : *Expansion of functions*

(i) $\tan^{-1}\left(\frac{1+x}{1-x}\right)$ (ii) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ (iii) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

(iv) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

These functions by the substitution $x = \tan\theta$ *respectively become*

(i) $\pi/4 + \tan^{-1} x$ (ii) $2\tan^{-1} x$ (iii) $1/2 \cdot \tan^{-1} x$ (iv) $2\tan^{-1} x$.

Hence the work executed in the Problem - 72 has to be carried out for completing the problem in the case of functions (i) to (iv)

Note - 2 : *Also in some cases we can venture to obtain a relation in* y, y_1, y_2 *and apply Leibnitz theorem to differentiate the result n times to obtain a relation in terms of* y_{n+2}, y_{n+1}, y_n. *By knowing* $y(0)$ *and* $y_1(0)$ *we can easily obtain* $y_2(0)$, $y_3(0)$, ... *by taking* $n = 0$, $n = 1, \ldots$

For example in the case of $y = \tan^{-1} x$ *we have the relation* [*Refer Problem* -13]

$$(1+x^2)\,y_{n+2} + 2(n+1)\,x\,y_{n+1} + n(n+1)\,y_n = 0$$

At $x = 0$, $y_{n+2}(0) = -n(n+1)\,y_n(0)$ ...(1)

We have $y(0) = 0$, $y_1(0) = 1$ *where* $y = \tan^{-1} x$.

Putting $n = 0, 1, 2, 3 \ldots$ *in the relation (1) we get,* $y_2(0) = 0$,

$$y_3(0) = -2\,y_1(0) = -2\ ,\ y_4(0) = -6\,y_2(0) = 0$$

$$y_5(0) = -12\,y_3(0) = (-12)(-2) = 24 \quad etc.$$

Note - 3 : *We can adopt this method for Problem - 59, where* $y = \sin^{-1} x$

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73. *Show that* $\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2 \cdot 4}{3 \cdot 5}x^5 + \ldots$

>> Let $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ $\therefore$ $y(0) = 0$

$$\sqrt{1-x^2}\,y = \sin^{-1} x$$

Differentiating w.r.t. x we get,

$$\sqrt{1-x^2}\cdot y_1 + \frac{1}{2\sqrt{1-x^2}}(-2x)\,y = \frac{1}{\sqrt{1-x^2}}$$

or $(1-x^2)y_1 - xy = 1$; At $x = 0$, $y_1(0) = 1$

Differentiating w.r.t. x again we get,

$$(1-x^2)y_2 - 2xy_1 - xy_1 - y = 0$$

or $(1-x^2)y_2 - 3xy_1 - y = 0$; At $x = 0$, $y_2(0) = 0$

Applying Leibnitz theorem we have,

$$\left\{(1-x^2)y_{n+2} + n\cdot(-2x)y_{n+1} + \frac{n(n-1)}{1\cdot 2}(-2)y_n\right\} - 3\left\{xy_{n+1} + n\cdot y_n\right\} - y_n = 0$$

ie., $(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - n^2 y_n + ny_n - 3ny_n - y_n = 0$

ie., $(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0$

At $x = 0$: $y_{n+2}(0) = (n+1)^2 y_n(0)$

$\therefore\ y_3(0) = 4\cdot y_1(0) = 4$; $y_4(0) = 9\cdot y_2(0) = 0$ $y_5(0) = 16\cdot y_3(0) = 64$ etc.

We have the expansion,

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \dots$$

ie., $$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x\cdot 1 + \frac{x^3}{6}\cdot 4 + \frac{x^5}{120}\cdot 64 + \dots = x + \frac{2}{3}x^3 + \frac{8}{15}x^5 + \dots$$

Thus $$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 + \dots$$

Remark : *We can as well apply Leibnitz theorem after finding a relationship involving* y_1 *so that we obtain a relationship involving* y_{n+1}

>> $$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \dots$$

Let $y = e^{a\sin^{-1}x}$ $\therefore\ y(0) = 1$

$\therefore$ $y_1 = e^{a\sin^{-1}x}\cdot\frac{a}{\sqrt{1-x^2}}$ or $y_1 = \frac{ay}{\sqrt{1-x^2}}$ $\therefore\ y_1(0) = a$

Consider $\sqrt{1-x^2}\, y_1 = a\,y$ and by differentiating w.r.t. x we get,

$$\sqrt{1-x^2}\, y_2 + \frac{-2x}{2\sqrt{1-x^2}}\, y_1 = a\, y_1$$

ie., $\quad (1-x^2)\, y_2 - x\, y_1 = a\sqrt{1-x^2}\, y_1 = a\,(a\,y)$

ie., $\quad (1-x^2)\, y_2 - x\, y_1 = a^2\, y$

Applying Leibnitz theorem we have,

$$\left\{(1-x^2)\, y_{n+2} + n\,(-2x)\, y_{n+1} + \frac{n\,(n-1)}{1\cdot 2}\,(-2)\, y_n\right\} - \left\{x y_{n+1} + n\cdot 1\cdot y_n\right\} = a^2\, y_n$$

ie., $\quad (1-x^2)\, y_{n+2} - (2n+1)\, x y_{n+1} - (n^2+a^2)\, y_n = 0$

At $x = 0$, $y_{n+2}(0) = (n^2+a^2)\, y_n(0)$

$\therefore \quad y_2(0) = a^2\, y(0) = a^2$; $y_3(0) = (1+a^2)\, y_1(0) = (1+a^2)\, a$

$y_4(0) = (4+a^2)\, y_2(0) = (4+a^2)\, a^2$

$y_5(0) = (9+a^2)\, y_3(0) = (9+a^2)\,(1+a^2)\, a$ etc.

Thus by substituting these values in the expansion of $y(x)$ we have,

$$e^{a\sin^{-1}x} = 1 + ax + a^2\frac{x^2}{2!} + \frac{a(1+a^2)}{3!}x^3 + \frac{a^2(4+a^2)}{4!}x^4 + \frac{a(9+a^2)\,(1+a^2)}{5!}x^5 + \ldots$$

---

75. *Obtain the Maclaurin's expansion of $e^{\tan^{-1}x}$ up to the term containing x^5.*

\>> $\quad y(x) = y(0) + x\, y_1(0) + \frac{x^2}{2!}\, y_2(0) + \ldots$

Let $\quad y = e^{\tan^{-1}x} \qquad \therefore \quad y(0) = 1$

$$y_1 = \frac{e^{\tan^{-1}x}}{1+x^2} \quad ie.,\ y_1 = \frac{y}{1+x^2} \qquad \therefore \quad y_1(0) = 1$$

Consider $(1+x^2)\,y_1 = y$ and by applying Leibnitz theorem we have,

$$(1+x^2)\,y_{n+1} + n \cdot 2x \cdot y_n + \frac{n(n-1)}{1\cdot 2}\cdot 2\,y_{n-1} = y_n$$

ie., $$(1+x^2)\,y_{n+1} + (2nx-1)\,y_n + n(n-1)\,y_{n-1} = 0$$

At $x = 0$, $y_{n+1}(0) = y_n(0) - n(n-1)\,y_{n-1}(0)$.

Hence we have,

$$y_2(0) = y_1(0) - 0 = 1$$

$$y_3(0) = y_2(0) - 2\,y_1(0) = 1 - 2 = -1$$

$$y_4(0) = y_3(0) - 6\,y_2(0) = -1 - 6 = -7$$

$$y_5(0) = y_4(0) - 12\,y_3(0) = -7 + 12 = 5$$

Substituting these values in the expansion of $y(x)$ we have,

$$e^{\tan^{-1}x} = 1 + x\cdot 1 + \frac{x^2}{2}\cdot 1 + \frac{x^3}{6}\cdot(-1) + \frac{x^4}{24}(-7) + \frac{x^5}{120}\cdot 5\ldots$$

Thus $$\mathbf{e^{\tan^{-1}x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7}{24}x^4 + \frac{x^5}{24}\ldots}$$

Note: *Maclaurin's expansions of the functions* $\sin x$, $\cos x$, $\sin h\, x$, $\cosh x$, e^x *can be found easily and it is advisible to remember them. They are as follows.*

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ (ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(iii) $\sin h\, x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$ (iv) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

(v) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

>> $$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \ldots$$

We have $f(x) = y = x/\sin x$. Here $y(0)$ assumes $0/0$ form and we know that

$\lim_{x\to 0} \frac{x}{\sin x} = 1 \quad \therefore \quad y(0) = 1$

y_1 also assumes $0/0$ form and we have to apply L' Hospital's rule to find $y_1(0)$. The process becomes highly tedious as we proceed. Hence we try to make use of the expansion of $\sin x$ and since $\sin x$ is in the denominator we cannot carry out any simplification. Hence we take the reciprocal of y and proceed as follows.

$$y = \frac{x}{\sin x} \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Now $$\frac{1}{y} = \frac{\sin x}{x} = \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right]$$

ie., $$\frac{1}{y} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

Differentiating w.r.t. x we have,

$$\frac{-1}{y^2} y_1 = \frac{-2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \dots \qquad \dots (1)$$

At $x = 0$, $-y_1(0) = 0$ since $y(0) = 1$ $\therefore$ $y_1(0) = 0$.

Differentiating (1) w.r.t. x we have,

$$\frac{-1}{y^2} y_2 + \frac{2}{y^3} y_1^2 = \frac{-2}{3!} + \frac{12x^2}{5!} - \frac{30x^4}{7!} + \dots \qquad \dots (2)$$

At $x = 0$, $-y_2(0) + 0 = -1/3$ $\therefore$ $y_2(0) = 1/3$

Differentiating (2) w.r.t. x we have,

$$\frac{-1}{y^2} y_3 + \frac{2}{y^3} y_1 y_2 + \frac{4}{y^3} y_1 y_2 - \frac{6}{y^4} y_1^3 = \frac{24x}{5!} - \frac{120x^3}{7!} + \dots$$

ie., $$-\frac{1}{y^2} y_3 + \frac{6}{y^3} y_1 y_2 - \frac{6}{y^4} y_1^3 = \frac{x}{5} - \frac{x^3}{42} + \dots \qquad \dots (3)$$

At $x = 0$, $y_3(0) = 0$

Differentiating (3) w.r.t. x we have, $\therefore$ $y_3(0) = 0$

$$\frac{-1}{y^2} y_4 + \frac{2}{y^3} y_1 y_3 + \frac{6y_1}{y^3} y_3 + \frac{6y_2}{y^3} y_2 + 6y_1 y_2 \left(\frac{-3}{y^4}\right) y_1$$

$$-\frac{6}{y^4}(3y_1^2 y_2) - 6y_1^3 \left(\frac{-4}{y^5} y_1\right) = \frac{1}{5} - \frac{x^2}{14} + \dots$$

At $x = 0$, $-y_4(0) + 6(1/9) = 1/5$

or $y_4(0) = 2/3 - 1/5 = 7/15$ $\therefore$ $y_4(0) = 7/15$

Substituting these values in the expansion of $y(x)$ we get,

$$\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$$

>> We have Maclaurin's series

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots \qquad \ldots(1)$$

By data $f(x) = y(x) = \dfrac{x}{e^{x-1}}$; $y(0) = 0$

i.e., $y = \dfrac{x}{e^x \cdot e^{-1}} = e \cdot \dfrac{x}{e^x}$ or $e^x y = e\,x$

We differentiate this equation successively four times and evaluate at $x = 0$ as follows.

$e^x y_1 + e^x y = e\,;\; 1 \cdot y_1(0) + 1 \cdot 0 = e$ $\therefore$ $y_1(0) = e$

$e^x y_2 + 2e^x y_1 + e^x y = 0;\; y_2(0) + 2e = 0$ $\therefore$ $y_2(0) = -2e$

$e^x y_3 + 3e^x y_2 + 3e^x y_1 + e^x y = 0\,;\; y_3(0) - 6e + 3e = 0$ $\therefore$ $y_3(0) = 3e$

$e^x y_4 + 4e^x y_3 + 6e^x y_2 + 4e^x y_1 + e^x y = 0$ $\therefore$ $y_4(0) = -4e$

Thus by substituting these values in (1) we have,

$$\frac{x}{e^{x-1}} = e\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \cdots\right)$$

Note : *If* $y = f(x)$ *is of the form* $\dfrac{x}{e^x - 1}$ *then* $y(0)$ *becomes* $\dfrac{0}{0}$ *which is indeterminate. We have to use the Maclaurin's series of* e^x *given by* $1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ *so that*

$$y(x) = \frac{x}{x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots} = \frac{1}{1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \cdots}\ ;\ y(0) = 1$$

We consider $\dfrac{1}{y} = 1 + \dfrac{x}{2} + \dfrac{x^2}{6} + \dfrac{x^3}{24} + \cdots$ *and proceed to differentiate successively.*
(Similar to the previous problem).

1. Expand $\tan x$ about the point $x = \pi/4$ upto the third degree terms and hence find $\tan 46°$

2. Show that
$\sqrt{x} = \sqrt{2}\,[\,1 + (x-2)/4 - (x-2)^2/32 + (x-2)^3/128 \ldots\,]$

Obtain the Maclaurin's expansion of the following functions as indicated

3. $e^{\cos x}$ upto the fourth degree terms.

4. $\log\ (\sec x + \tan x)$ upto the first three non vanishing terms.

5. $\log(1 + \sin^2 x)$ upto the fourth degree terms.

6. $\log(1 + \tan x)$ upto the third degree terms.

7. $e^x \sin x$ upto the fifth degree terms.

8. Show that : $\log(x + \sqrt{x^2+1}) = x - \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1\cdot3}{2\cdot4}\dfrac{x^5}{5} - \dfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\dfrac{x^7}{7} + \ldots$
and hence find the value of $\log_e 2$ correct to two decimal places

9. Show that if $x = \sin t$, $y = \sin mt$ then
$y(x) = mx - \dfrac{m(m^2-1)}{3!}x^3 + \dfrac{m(m^2-1)(m^2-9)}{5!}x^5 - \ldots$

10. Show that $(\sin^{-1} x)^2 = \dfrac{2x^2}{2!} + \dfrac{2^2}{4!}2x^4 + \dfrac{2^2\cdot4^2}{6!}2x^6 + \ldots$

[**Hint** : *Use the* n^{th} *derivative approach for problems* 8 *to* 10.]

1. $1 + 2(x - \pi/4) + 2(x - \pi/4)^2 + (8/3)(x - \pi/4)^3$; 1 · 035

3. $e \left[1 - \frac{x^2}{2} + \frac{x^4}{6} \right]$

4. $x + x^3/6) + (x^5/24)$

5. $x^2 - (5x^4/6)$

6. $x - (x^2/2) + (2x^3/3)$

7. $x + x^2 + (x^3/3) - (x^5/30)$